Every nice graph is total weight $(1, 5)$-choosable

Xuding Zhu

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Total weighting

Edge weighting

An edge weighting f assigns to each edge e a real number $f(e)$ as its weight.

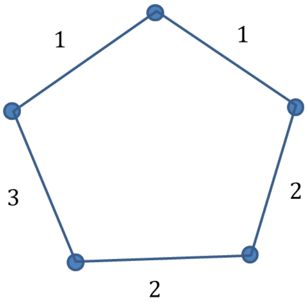

Figure: Edge weighting of C_5.

vertex-sum

The vertex sum at v is

$$S_f(v) = \sum_{e \in E(v)} f(e).$$

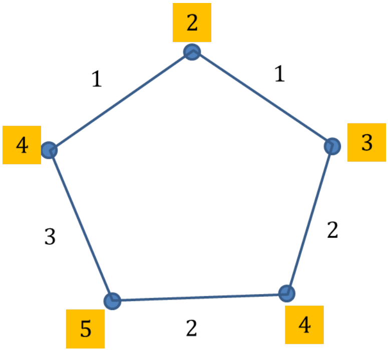

Figure: Corresponding vertex sums.

Proper

f is **proper** if for every edge $uv \in E$,

$$S_f(u) \neq S_f(v).$$

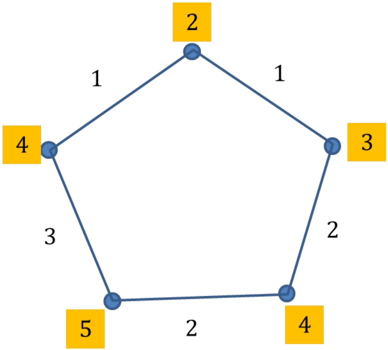

Figure: A proper edge-weighting.

Proper edge weighting using weights $1, 2, 3$

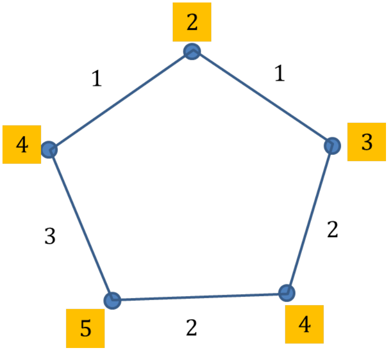

Figure: A proper edge-weighting.

Using weights $1, 2$ is not enough.

1-2-3 Conjecture

Observation

If G has an isolated edge, then G has no proper edge weighting.

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A graph is nice if it has no isolated edges.

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1-2-3 Conjecture

Every nice graph has a proper 3-edge weighting, i.e., using weights $1, 2, 3$.

Results on 1-2-3 Conjecture

Theorem [Karoński, Łuczak and Thomason (2004)]

There exists 183 real numbers so that every nice graph has a proper edge weighting using the 183 real numbers as weights.

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Every nice graph has a proper 30-edge weighting.

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Theorem [Addario-Berry, Dalal and Reed (2008)]

Every nice graph has a proper 16-edge weighting.

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Theorem [Wang and Yu (2008)]

Every nice graph has a proper 13-edge weighting.

Best result on 1-2-3 Conjecture

Best result [Kalkowski, Karoński and Pfender (2010)]

Every nice graph has a proper 5-edge weighting.

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$$f : E \to \{1, 2, 3, 4, 5\}.$$

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$$f : E \to \{1, 2, 3, 4, 5\}.$$

How about

$$f : E \to \{0, 2, 3, 4, 5\}?$$

Total weighting

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A total weighting f assigns to each $z \in V \cup E$ a real number $f(z)$ as its weight.

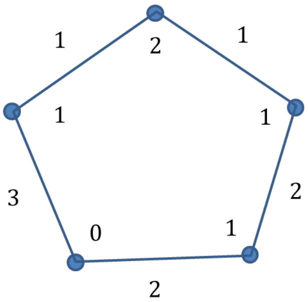

Figure: Total weighting of C_5.

Vertex-sum of a total weighting

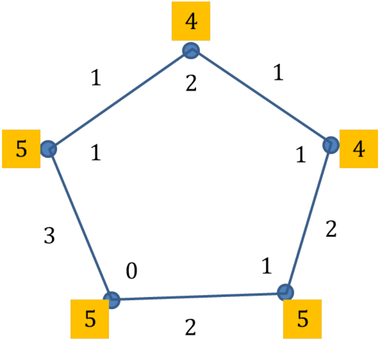

Figure: Corresponding vertex sums.

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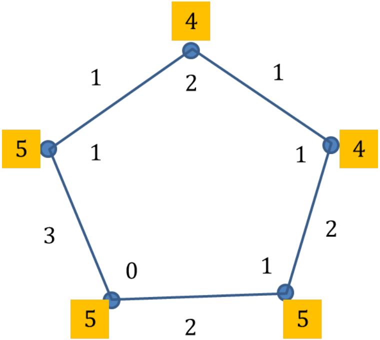

Figure: Not proper!

Using weights 1,2

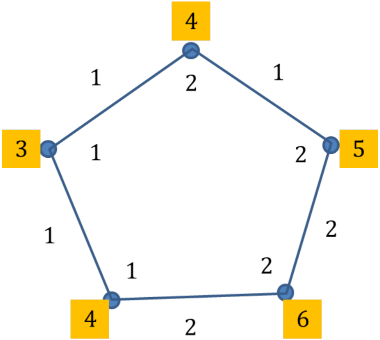

Figure: A proper total weighting of C_5, using weights $1, 2$.

1-2 Conjecture

1-2 Conjecture [Przybyło and Woźniak 2011]

Every graph has a proper total weighting using weights $1, 2$.

Best result on 1-2 conjecture

Best result [Kalkowski 2010]

Every graph has a proper total weighting f, such that $f(v) \in \{1, 2\}$ for $v \in V$ and $f(e) \in \{1, 2, 3\}$ for $e \in E$.

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Best result [Kalkowski 2010]

Every graph has a proper total weighting f, such that $f(v) \in \{1, 2\}$ for $v \in V$ and $f(e) \in \{1, 2, 3\}$ for $e \in E$.

How about $f(v) \in \{0, 2\}$ for $v \in V$ and $f(e) \in \{1, 2, 3\}$ for $e \in E$?

List version

List edge weighting

L assigns to each edge e a set $L(e)$ of **permissble weights.** A proper L-edge weighting is $f : E \to \mathbb{R}$ such that

$$f(e) \in L(e) \forall e \in E, \quad S_f(u) \neq S_f(v) \forall uv \in E.$$

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List total weighting

$L(z)$ for $z \in V \cup E$.

$$f(z) \in L(z) \forall z \in V \cup E, \quad S_f(u) \neq S_f(v) \forall uv \in E.$$

Conjectures

3-edge weight choosable conejcture [Bartnicki, Grytczuk and Niwczyk (2009)]

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$(2,2)$-choosable conjecture [Wong and Zhu (2011)]

Every graph is (2,2)-total weight choosable.

Weaker conjecture

[Wong and Zhu (2011)] There are constants k, k' such that

(A) every graph is (k, k')-choosable.

(B) every nice graph is $(1, k')$-choosable.

(C) every graph is $(k, 2)$-choosable.

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Theorem [Wong-Zhu (2012)]

Every graph is $(2, 3)$-choosable.

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Theorem [Wong-Zhu (2012)]

Every graph is $(2, 3)$-choosable.

Best result [Kalkowski 2010]

Every graph has a proper total weighting f, such that $f(v) \in \{1, 2\}$ for $v \in V$ and $f(e) \in \{1, 2, 3\}$ for $e \in E$.

Theorem [Chao (2021)]

Every nice graph is $(1, 17)$-choosable.

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Every nice graph has a proper edge weighting

$$f : E \to \{1, 2, 3, 4, 5\}.$$

But it was not known if we can choose

$$f : E \to \{0, 2, 3, 4, 5\}.$$

Proof

Theorem [Zhu (2021)]

Every nice graph is $(1,5)$-choosable.

$$\tilde{P}_G(\{x_z : z \in V \cup E\}) = \prod_{\{i,j\} \in E, i<j} \left(\left(\sum_{e \in E(i)} x_e + x_i \right) - \left(\sum_{e \in E(j)} x_e + x_j \right) \right).$$

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For $\phi : V \cup E \to \mathbb{R}$,

$$\tilde{P}_G(\phi) = \tilde{P}_G(\phi(e) : e \in E\}).$$

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ϕ is a proper total weighting iff $\tilde{P}_G(\phi) \neq 0$.

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$\tilde{P}_G$ is a homogenous polynomial.
All monomials are of the highest degree.

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$$|L(z)| \geq K(z) + 1$$

then G has a proper total L-weighting.

Definition

G is *algebraic total weight* (k, k')*-choosable* if

$$\exists x^K = \prod_{z \in V \cup E} x_z^{K(z)} \in \mathrm{mon}(\tilde{P}_G),$$

$$K(i) < k \ \ \forall i \in V, \ \ K(e) < k' \ \ \forall e \in E.$$

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Theorem [Wong-Zhu (2012)]

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$$x^K = \prod_{e \in E} x_e^{K(e)} \in \mathrm{mon}(\tilde{P}_G) \Leftrightarrow x^K = \prod_{e \in E} x_e^{K(e)} \in \mathrm{mon}(P_G).$$

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Definition

$K \in \mathbb{N}^E$ is **sufficient for** G if there exists $K' \in \mathbb{N}^E_{|E|}$ such that $K' \leq K$ and $x^{K'} \in \mathrm{mon}(P_G)$.

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Observation

G is algebraic $(1, b+1)$-choosable $\Leftrightarrow$
$\exists K \in \mathbb{N}^E$ such that K is sufficient for G and $K(e) \leq b$ for $e \in E$.

$$P_G(\{x_e : e \in E\}) = \prod_{e=\{i,j\}\in E, i<j} \left(\sum_{e' \in E(i)} x_{e'} - \sum_{e' \in E(j)} x_{e'} \right).$$

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$$P_G(\{x_e : e \in E\}) = \prod_{e \in E} \sum_{e' \in E} c_{ee'} x_{e'}.$$

$C_G = (c_{ee'})$ is an $|E| \times |E|$ matrix.

For $e = \{i, j\} \in E, i < j$,

$$c_{ee'} = \begin{cases} 1, & \text{if } e' \text{ is adjacent with } e \text{ at } i, \\ -1, & \text{if } e' \text{ is adjacent with } e \text{ at } j, \\ 0, & \text{otherwise.} \end{cases}$$

Given a matrix

$$A = (a_{ij})_{m\times n},$$

$$F_A(x_1, \ldots, x_n) = \prod_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_j.$$

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F_A and $A = (a_{ij})_{m\times n}$

All information about the polynomial F_A is contained in the matrix A.

Coefficient of x^K in F_A

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

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$$\begin{aligned} F_A = \; & (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n) \\ & (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n) \\ & \cdots\cdots \\ & (a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n). \end{aligned}$$

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$$\mathrm{coe}(x_1^m, F_A) = a_{1,1}a_{2,1}\ldots a_{m,1}$$

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$$\begin{aligned} \mathrm{coe}(x_1^{m-1}x_2, F_A) &= a_{1,2}a_{2,1}\dots a_{m,1} \\ &+ a_{1,1}a_{2,2}\dots a_{m,1} \\ &\quad \dots\dots \\ &+ a_{1,1}a_{2,1}\dots a_{m,2}. \end{aligned}$$

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$$\mathrm{coe}(x^K, F_A) = \mathrm{coe}(x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}, F_A) = \cdots\cdots$$

$$\mathrm{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

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$$\mathrm{coe}(x^K, F_A) = \frac{1}{K!}\mathrm{per}(A(K)).$$

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$$A_G = (a_{ei})_{e \in E, i \in V},$$

where for $e = \{s, t\} \in E, s < t$,

$$a_{ei} = \begin{cases} 1, & \text{if } i = s, \\ -1, & \text{if } i = t, \\ 0, & \text{otherwise.} \end{cases}$$

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$$C_G = A_G B_G^T,$$

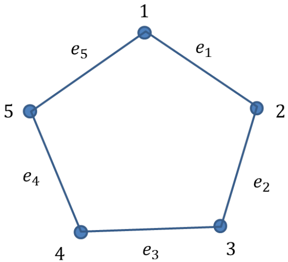

Figure: $G = C_5$.

$$A_G = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad B_G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Recall that

$C_K(K)$: The matrix consisting of $K(i)$ copies of the ith column of C_G.

$$\mathrm{coe}(x^K, P_G) = \frac{1}{K!}\mathrm{per}(C_G(K)) = \frac{1}{K!}\mathrm{per}(A_G B_G[K]^T).$$

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Assume $A, B \in M_{m,n}(\mathbb{C})$.

$$AB^* = (\overline{b_{11}}\mathrm{col}_1(A) + \ldots + \overline{b_{1n}}\mathrm{col}_n(A), \ldots, \overline{b_{m1}}\mathrm{col}_1(A) + \ldots + \overline{b_{mn}}\mathrm{col}_n(A)).$$

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For $\sigma \in [n]^m$,

$$M_\sigma = (\overline{b_{1\sigma(1)}}\mathrm{col}_{\sigma(1)}(A), \ldots, \overline{b_{m\sigma(m)}}\mathrm{col}_{\sigma(m)}(A)).$$

$$\mathrm{coe}(x^K, P_G) = \frac{1}{K!}\mathrm{per}(C_G(K)) = \frac{1}{K!}\mathrm{per}(A_G B_G[K]^T).$$

Assume $A, B \in M_{m,n}(\mathbb{C})$.

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For $\sigma \in [n]^m$,

$$M_\sigma = (\overline{b_{1\sigma(1)}}\mathrm{col}_{\sigma(1)}(A), \ldots, \overline{b_{m\sigma(m)}}\mathrm{col}_{\sigma(m)}(A)).$$

Permanent is linear with respect to its columns. So

$$\mathrm{per}(AB^*) = \sum_{\sigma\in[n]^m} \mathrm{per}(M_\sigma).$$

For $K = (k_i)_{i\in[n]} \in \mathbb{N}_m^n$, let

$$S(K) = \{\sigma \in [n]^m : |\sigma^{-1}(i)| = k_i\}.$$

For $\sigma \in S(K)$,

$$\mathrm{per}(M_\sigma) = \left(\prod_{j=1}^{m} \overline{b_{j\sigma(j)}}\right) \mathrm{per}(A(K)).$$

As $\sum_{\sigma\in S(K)} \prod_{j=1}^{m} \overline{b_{j\sigma(j)}} = \overline{\mathrm{coe}(x^K, F_B)}$,

$$\begin{aligned}\sum_{\sigma\in S(K)} \mathrm{per}(M_\sigma) &= \left(\sum_{\sigma\in S(K)} \prod_{j=1}^{m} \overline{b_{j\sigma(j)}}\right) \mathrm{per}(A(K)) \\ &= \overline{\mathrm{coe}(x^K, F_B)}\mathrm{per}(A(K)).\end{aligned}$$

$$\begin{aligned}\mathrm{per}(AB^*) &= \sum_{K\in\mathbb{N}_m^n} \overline{\mathrm{coe}(x^K, F_B)}\,\mathrm{per}(A(K)) \\ &= \sum_{K\in\mathbb{N}_m^n} K!\,\overline{\mathrm{coe}(x^K, F_B)}\,\mathrm{coe}(x^K, F_A).\end{aligned}$$

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with basis: $\{x^K : K \in \mathbb{N}^m_m\}$.

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Define an inner product in $\mathbb{C}[x_1, \ldots, x_n]_m$ as:

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$$\begin{aligned}\mathrm{coe}(x^K, P_G) &= \frac{1}{K!}\mathrm{per}(C_G(K)) \\ &= \frac{1}{K!}\langle F_{A_G}, F_{B_G[K]}\rangle.\end{aligned}$$

$A_G = (a_{ei})_{e \in E, i \in V}$, where for $e = \{s, t\} \in E, s < t$,

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$$F_{A_G} = \prod_{e=\{i,j\}\in E, i<j} (x_i - x_j) = Q_E.$$

$$F_{B_G[K]} = \prod_{e=\{i,j\}\in E} (x_i + x_j)^{K(e)} = H_E^K.$$

G is algebraic $(1, 5)$-choosable

$$\Longleftrightarrow$$

For some $K \in \mathbb{N}^{E}_{|E|}$, $K(e) \leq 4$ for $e \in E$,

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Definition

For $K \in \mathbb{N}^E$, let W_E^K be the complex linear space spanned by

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Lemma

Assume $G = (V, E)$ and $K \in \mathbb{N}^E$. The following are equivalent:

1. K is sufficient for G.
2. $\langle H_E^{K'}, Q_E \rangle \neq 0$ for some $K' \leq K$.
3. $\langle F, Q_E \rangle \neq 0$ for some $F \in W_E^K$.
4. $\mathrm{per}(C_G(K')) \neq 0$ for some $K' \leq K$.

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Subgraphs associated to a subset J of V

Find $F \in W_E^K$ such that

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F, Q_E are polynomials in $\{x_i : i \in V(G)\}$.

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Observation

If $F = F_1 F_2$ and $E = E_1 \cup E_2$ (and hence $Q_E = Q_{E_1} Q_{E_2}$), and F_1 and F_2, Q_{E_2} have no variables in common, F_2 and Q_{E_1} have no variables in common, then

$$\langle F, Q_E \rangle = \langle F_1, Q_{E_1} \rangle \langle F_2, Q_{E_2} \rangle.$$

Definition

Assume $G = (V, E)$ is a graph and J is a subset of V.

1. $E_{J,1}$ the set of non-isolated edges in $G - J$.
2. $E_{J,2}$ the set of isolated edges in $G - J$.
3. $E_{J,3}$ the set of edges with exactly one end vertex in J.
4. $E_{J,4}$ the set of edges with both end vertices in J.

$G_{J,i}$ be the subgraph induced by $E_{J,i}$.

Subgraphs associated to a subset J of V

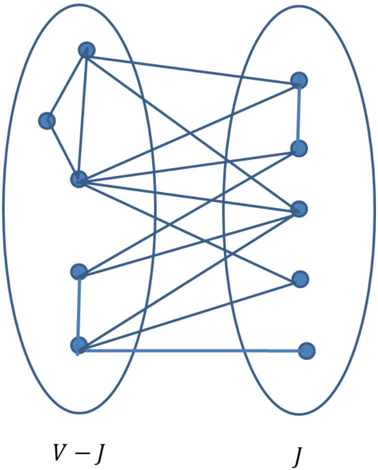

Figure: A subset J of $V(G)$.

Subgraphs associated to a subset J of V

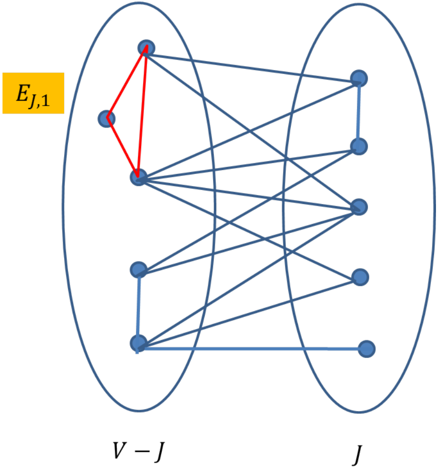

Figure: Edge set $E_{J,1}$.

Subgraphs associated to a subset J of V

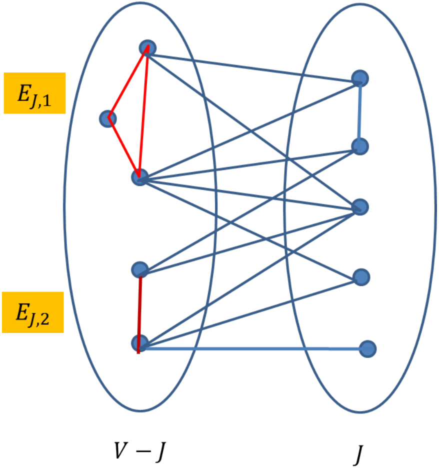

Figure: Edge set $E_{J,2}$.

Subgraphs associated to a subset J of V

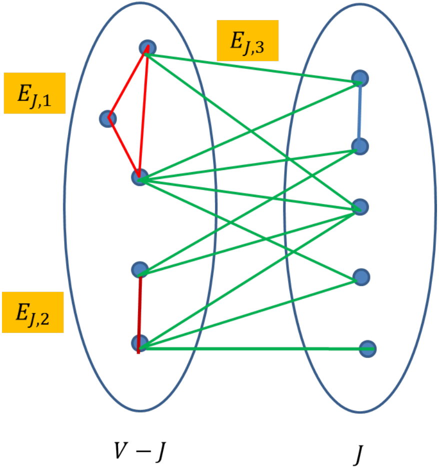

Figure: Edge set $E_{J,3}$.

Subgraphs associated to a subset J of V

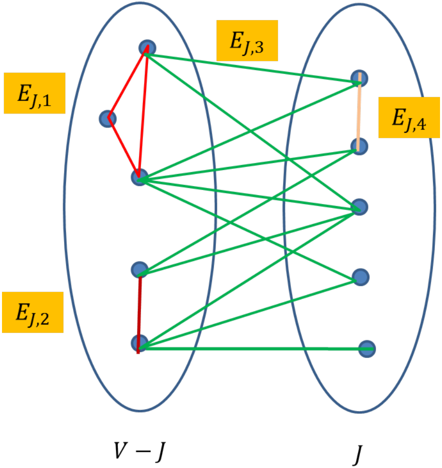

Figure: Edge set $E_{J,4}$.

An $E_{J,2}$-covering family

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C_e: an edge in $E_{J,3}$ adjacent to e.

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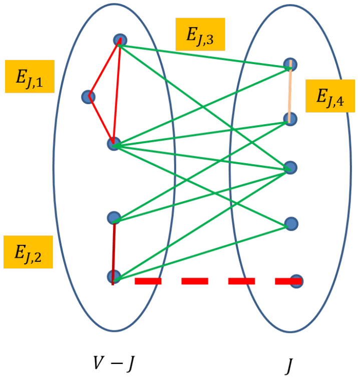

Figure: An $E_{J,2}$-covering family.

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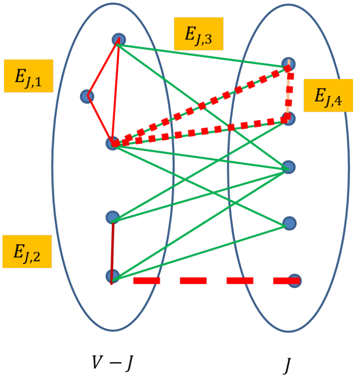

Figure: An $E_{J,4}$-covering family.

An $E_{J,3}$-covering family $\mathcal{C}_{J,3}$

- For $i, j \in J$, an even number of even length i-j-paths; and an even number of odd length i-j-paths.
- For each $i \in J$, $d_{\mathcal{E}_{j,4}}(i) \geq 2d_{G_{J,3}}(i)$.

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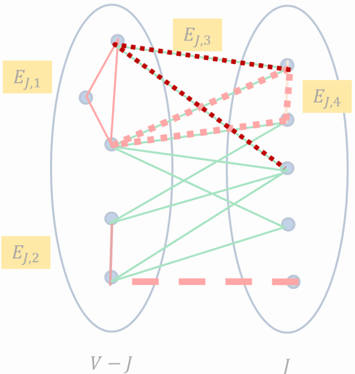

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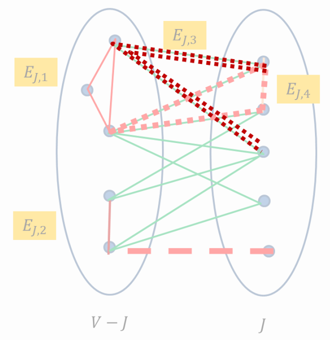

A J-covering family $\mathcal{C}$

$$\mathcal{C} = \mathcal{C}_{J,2} \cup \mathcal{C}_{J,3} \cup \mathcal{C}_{J,4}.$$

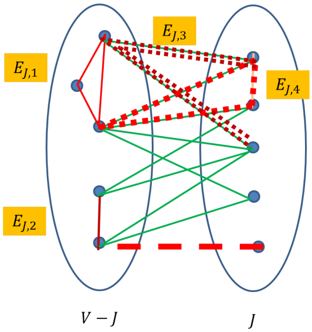

Figure: Part of a J-covering family.

For a graph H of G, $K_H : E \to \mathbb{N}$ is defined as

$$K_H(e) = \begin{cases} 1, & \text{if } e \in E(H) \\ 0, & \text{Otherwise.} \end{cases}$$

$$K_{\mathcal{C}} = \sum_{H \in \mathcal{C}} K_H.$$

$K_{\mathcal{C}}(e) =$ number of subgraphs in $\mathcal{C}$ containing e.

Key Lemma

Assume $G = (V, E)$ is a graph, and J is a subset of V, $\mathcal{C}$ is a J-covering family. If $K \in \mathbb{N}^E$ is sufficient for $G_{J,1}$, then $K + K_{\mathcal{C}}$ is sufficient for G.

Key Lemma

Assume $G = (V, E)$ is a graph, and J is a subset of V, $\mathcal{C}$ is a J-covering family. If $K \in \mathbb{N}^E$ is sufficient for $G_{J,1}$, then $K + K_{\mathcal{C}}$ is sufficient for G.

Corollary

If $G_{J,1}$ is algebraic $(1, b+1)$-choosable, and there exists a J-covering family $\mathcal{C}$ such that $K_{\mathcal{C}}(e) \leq b$ for $e \in E - E_{J,1}$ and $K_{\mathcal{C}}(e) = 0$ for $e \in E_{J,1}$, then G is algebraic $(1, b+1)$-choosable.

Proof by induction

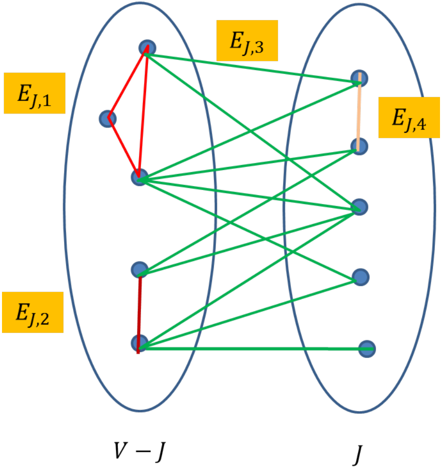

Figure: $K + K_{\mathcal{C}}$ is sufficient for G.

Theorem

Every nice graph is $(1,5)$-choosable.

***J*-cover Lemma**

There is a subset J, and a J-covering family $\mathcal{C}$ such that $K_{\mathcal{C}}(e) \leq 4$ for $e \in E - E_{J,1}$ and $K_{\mathcal{C}}(e) = 0$ for $e \in E_{J,1}$.

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By induction hypothesis, $G_{J,1}$ is algebraic $(1,5)$-choosable. Some K is sufficient for $G_{J,1}$, $K(e) \leq 4$ for $e \in E_{J,1}$, and $K(e) = 0$ for $e \notin E_{J,1}$.

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$(K + K_{\mathcal{C}})(e) \leq 4$ for all e.

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$(K + K_{\mathcal{C}})(e) \leq 4$ for all e.
$K + K_{\mathcal{C}}$ is sufficient for G.
So G is $(1, 5)$-choosable.

Proof of a weaker version of J-cover Lemma

Weaker version J-cover Lemma

There is a subset J, and a J-covering family $\mathcal{C}$ such that $K_{\mathcal{C}}(e) \leq 5$ for $e \in E - E_{J,1}$ and $K_{\mathcal{C}}(e) = 0$ for $e \in E_{J,1}$.

Proof of a weaker version of J-cover Lemma

Weaker version J-cover Lemma

There is a subset J, and a J-covering family $\mathcal{C}$ such that $K_{\mathcal{C}}(e) \leq 5$ for $e \in E - E_{J,1}$ and $K_{\mathcal{C}}(e) = 0$ for $e \in E_{J,1}$.

Definition

$J \subseteq V(G)$ is good, if

- $G_{J,4}$ has maximum degree ≤ 1.
- Each vertex $i \in J$ has at most 1 private neighbour in $G_{J,3}$.
- If ij is an edge in $G_{J,4}$, then none of i, j has a private neighbour and i, j have a common neighbour.

Lemma

There is a good subset J.

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There is a good subset J.

Choose a maximum independent set J of G

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If $j \in J$ has more than one private neighbours,

then the set S of private neighbours of j induces a clique.

Move one vertex of S to J.

Repeat this for each $j \in J$ with more than one private neighbours, we obtain a nice subset J.

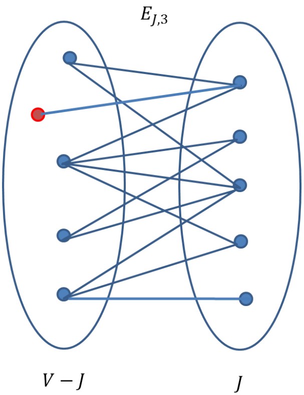


Figure: Private neighbor.

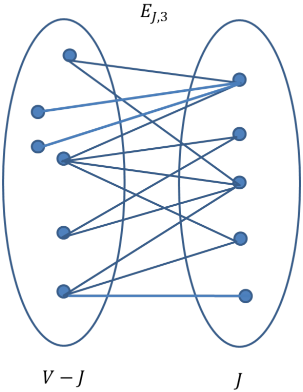
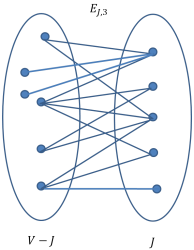


Figure: j has more than 1 private neighbor.



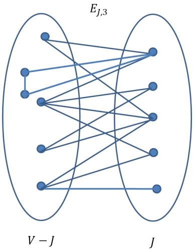


Figure: Private neighbors of a vertex form a clique

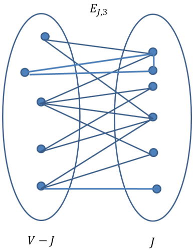


Figure: Move one private neighbor to J

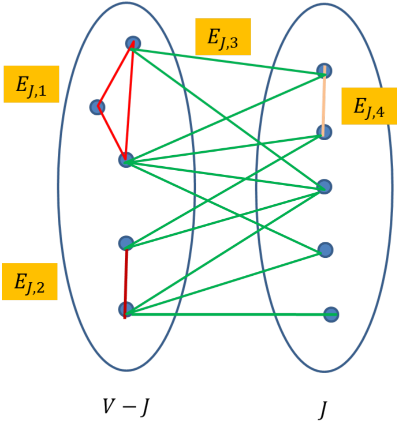


Figure: This set J is good.

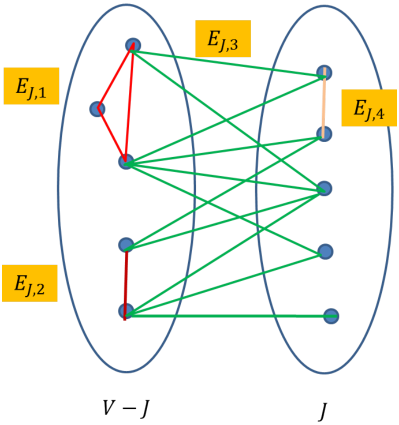


Figure: This set J is good.

If J is good, then there is a J-covering family $\mathcal{C}$ such that $K_{\mathcal{C}}(e) \leq 5$ for each e and $K_{\mathcal{C}}(e) = 0$ for $e \in E_{J,1}$.

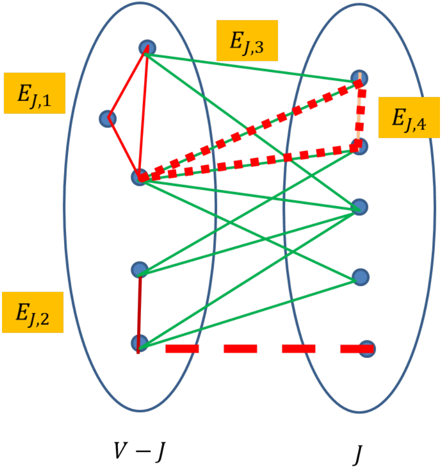


Figure: $E_{J,2}$-cover and $E_{J,4}$-cover.

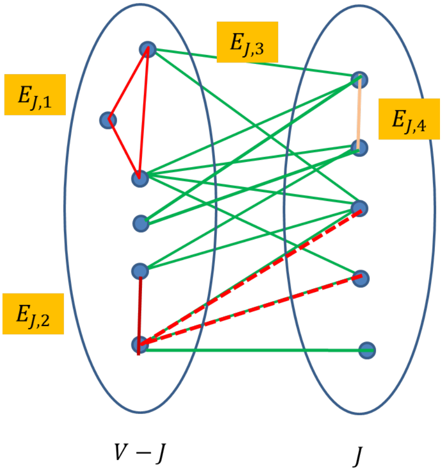
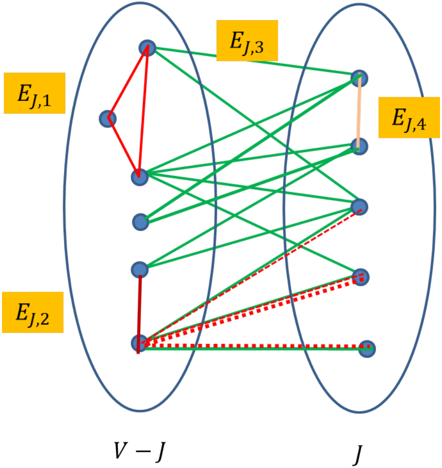


Figure: One path in $\mathcal{C}_{J,3}$.



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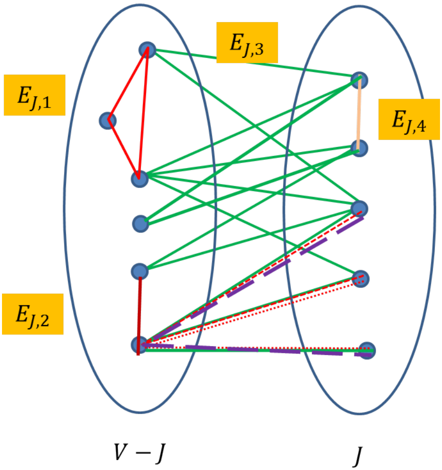

Figure: Three paths in $\mathcal{C}_{J,3}$ incident to a vertex $v \in V - J$ of degree 3 in $G_{J,3}$.

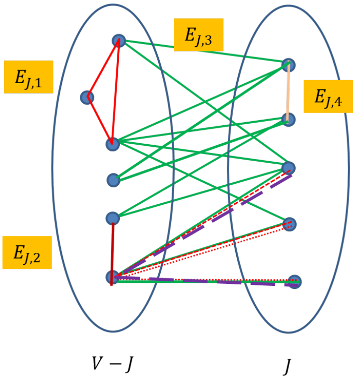

Figure: **3** paths incident to a vertex $v \in V - J$ of degree **3** in $G_{J,3}$.

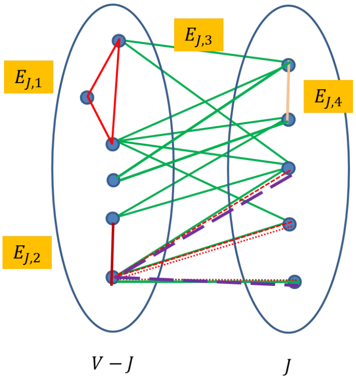

Figure: **3** paths incident to a vertex $v \in V - J$ of degree **3** in $G_{J,3}$.

Double each of the **3** paths.

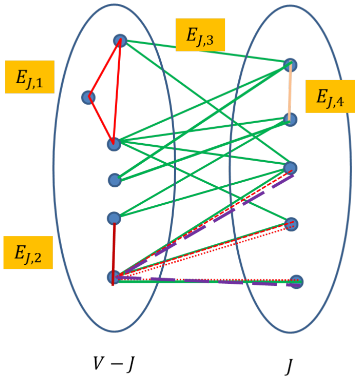

Figure: **3** paths incident to a vertex $v \in V - J$ of degree **3** in $G_{J,3}$.

Double each of the **3** paths.
Each vertex $v \in V - J$ of degree **d** contributes **2d** paths to $\mathcal{C}_{J,3}$.

Assume

$$e = \{i, j\} \in E_{J,3}, i \in V - J, j \in J.$$

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If i is not a private neighor of j, then e contributes
4 to $d_{\mathcal{C}_{J,3}}(j)$
1 to $d_{G_{J,3}}(j)$.

If i is a private neighor of j, then e contributes
0 to $d_{\mathcal{C}_{J,3}}(j)$
1 to $d_{G_{J,3}}(j)$.

As j has at most one private neighbor, and at least one non-private neighbor,

$$d_{\mathcal{C}_{J,3}}(j) \geq 2d_{G_{J,3}}(j).$$

Each edge in $E_{J,3}$ is contained in at most 4 paths in $\mathcal{C}_{J,3}$,

At most 1 edge in $\mathcal{C}_{J,2}$,

At most 1 odd cycle in $\mathcal{C}_{J,4}$.

Moreover, a little care shows that if $e \in E_{J,3}$ is contained in $\mathcal{C}_{J,4}$, then it is contained in at most 2 paths in $\mathcal{C}_{J,3}$.

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So $K_{\mathcal{C}}(e) \leq 5$ for each edge e,
$K_{\mathcal{C}}(e) = 0$ if $e \in E_{J,1}$.

Proof of Key Lemma

Key Lemma

Assume $G = (V, E)$ is a graph, and J is a subset of V, $\mathcal{C}$ is a J-covering family. If $K \in \mathbb{N}^E$ is sufficient for $G_{J,1}$, then $K + K_{\mathcal{C}}$ is sufficient for G.

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Need to find $F \in W_E^{K+K_{\mathcal{C}}}$ so that

$$\langle F, Q_E \rangle \neq 0.$$

$$E = E_{J,1} \cup E_{J,2} \cup E_{J,3} \cup E_{J,4}.$$

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We shall find

$$\begin{aligned} F_1 &\in W_E^K, \\ F_2 &\in W_E^{K_{\mathcal{C}_{J,2}}}, \\ F_3 &\in W_E^{K_{\mathcal{C}_{J,3}}}, \\ F_2 &\in W_E^{K_{\mathcal{C}_{J,4}}}, \end{aligned}$$

$$F = F_1 F_2 F_3 F_4 \in W_E^{K + K_{\mathcal{C}}}.$$

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Assume $G = (V, E)$ is a graph, and J is a subset of V, $\mathcal{C}$ is a J-covering family. If $K \in \mathbb{N}^E$ is sufficient for $G_{J,1}$, then $K + K_{\mathcal{C}}$ is sufficient for G.

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By hypothesis,

$$\exists F_1 \in W_E^K, \langle F_1, Q_{E_1} \rangle \neq 0.$$

For each edge $e = \{i, j\}$ of G, let

$$F_e = x_i + x_j.$$

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$$F_2 = \prod_{e \in E_{J,2}} F_{C_e} \in W_E^{K_{\mathcal{C}_{J,2}}}.$$

For $P \in \mathcal{C}_{J,3}$ connecting $i, j \in J$ with $i < j$, let

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Let $\ell(P)$ be the length of P.

For $l = 1, \ldots, \ell(P)$, let $e_l(P)$ be the lth edge of P.

$$\begin{aligned} F_3 &= \prod_{P \in \mathcal{C}_{J,3}} (x_{s(P)} + (-1)^{\ell(P)-1} x_{t(P)}) \\ &= \prod_{P \in \mathcal{C}_{J,3}} \sum_{l=1}^{\ell(P)} (-1)^{l-1} F_{e_l(P)} \in W_E^{K_{\mathcal{C}_{J,3}}} \end{aligned}$$

lemma

For $e = \{j_1, j_2\} \in E_{J,4}$, $x_{j_1}, x_{j_2} \in W_E^{K_{C_e}}$.

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Then

$$F_4 = \prod_{e \in E_{J,4}} x_{j_e} = \prod_{e \in E_{J,4}} \frac{1}{2} \sum_{l=1}^{\ell(C_e)} (-1)^{l-1} F_{e_l(C_e)} \in W_E^{K_{\mathcal{C}_{J,3}}}.$$

1. $\exists F_1 \in W_E^K, \langle F_1, Q_{E_1} \rangle \neq 0.$
2. $F_2 = \prod_{e \in E_{J,2}, C_e = \{i_e, j_e\} \in \mathcal{C}_{J,2}} (x_{i_e} + x_{j_e}) \in W_E^{K_{\mathcal{C}_{J,2}}}.$
3. $F_3 = \prod_{P \in \mathcal{C}_{J,3}} (x_{s(P)} + (-1)^{\ell(P)-1} x_{t(P)}) \in W_E^{K_{\mathcal{C}_{J,3}}}.$
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It remains to show that

$$\langle F_1 F_2 F_3 F_4, Q_{E_{J,1}} Q_{E_{J,2}} Q_{E_{J,3}} Q_{E_{J,4}} \rangle \neq 0.$$

F_1 and $Q_{E_{J,1}}$ are polynomials in variables x_i for $i \in V(G_{J,1})$.

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By assumption, $\langle F_1, Q_{E_{J,1}} \rangle \neq 0$.
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Hence

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We can replace F_2 by

$$\prod_{e \in E_{J,2}} x_{i_e}.$$

$$\begin{aligned}\langle F_2F_3F_4, Q_{E_{J,2}}Q_{E_{J,3}}Q_{E_{J,4}}\rangle &= \langle \prod_{e\in E_{J,2}} x_{i_e}, Q_{E_{J,2}}\rangle\langle F_3F_4, Q_{E_{J,3}}Q_{E_{J,4}}\rangle \\ &= \pm\langle F_3F_4, Q_{E_{J,3}}Q_{E_{J,4}}\rangle\end{aligned}$$

The monomial $\prod_{e\in E_{J,2}} x_{i_e}$ has coefficient 1 or -1 in $Q_{E_{J,2}}$.

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Then

$$F_3 = \prod_{i,j \in L, i<j} (x_i - x_j)^{2t_{ij}^{+}} (x_i + x_j)^{2t_{ij}^{-}} \in W_E^{K_{\mathcal{C}_{J,3}}}.$$

F_3, F_4 and $Q_{E_J,4}$ are polynomials in variables x_i for $i \in J$.

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In the expansion of $Q_{E_{J,3}} Q_{E_{J,4}}$, we only need to consider those monomials in which $Q_{E_{J,3}}$ only contributes to variables x_i for $i \in J$.

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$$\prod_{e=\{i_e,j_e\}\in E_{J,3}} x_{j_e} = \prod_{i,j,\in J, i<j} (x_i x_j)^{t_{ij}^{+}+t_{ij}^{-}}.$$

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By adding some arbitrary edges to $G_{J,3}$, we may assume that

$$d_{\mathcal{C}_{J,3}}(i) = 2d_{G_{J,3}}(i).$$

$$\prod_{e=\{i_e,j_e\}\in E_{J,3}} x_{j_e} = \prod_{i,j,\in J, i<j} (x_i x_j)^{t_{ij}^+ + t_{ij}^-}.$$

Let

$$\phi = \prod_{i,j\in L, i<j} (x_i - x_j)^{2t_{ij}^+}(x_i + x_j)^{2t_{ij}^-},\ \psi = \prod_{i,j,\in J, i<j} (x_i x_j)^{t_{ij}^+ + t_{ij}^-}.$$

We need to show that

$$\langle \phi F_4, \psi Q_{E_{J,4}} \rangle \neq 0.$$

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Recall that

$$F_4 = \prod_{e \in E_{J,4}} x_{j_e} \in \mathrm{mon}(Q_{E_{J,4}}).$$

It suffices to show

Lemma 1

For any nonzero polynomial $R(x) \in \mathbb{C}[x_1, \ldots, x_n]$,
$\exists x^K \in \mathrm{mon}(R(x))$

$$\langle \phi x^K, \psi R(x) \rangle \neq 0.$$

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Assume to the contrary that for each $1 \leq i \leq l$,

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For $1 \leq i \leq l$, $\psi(x) x^{K_i}$ is also a monomial $x^{\tilde{K}_i}$.

$$\mathrm{mon}(\psi(x) R(x)) = \{x^{\tilde{K}_i} : i = 1, \ldots, l\}.$$

Recall the inner product in $\mathbb{C}[x_1, x_2, \ldots, x_n]_m$ is defined as

$$\langle f, g \rangle = \sum_{K \in \mathbb{N}_m^n} K! \mathrm{coe}(x^K, f) \overline{\mathrm{coe}(x^K, g)}.$$

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We define another inner product

$$(f, g) = \sum_{K \in \mathbb{N}_m^n} \mathrm{coe}(x^K, f)\overline{\mathrm{coe}(x^K, g)}, \forall f, g \in \mathbb{C}[x_1, \ldots, x_n]_m.$$

Lemma A

Assume $f, g \in \mathbb{C}[x_1, \ldots, x_n]_m$. Let $\tilde{f} \in \mathbb{C}[x_1, \ldots, x_n]_m$ be a polynomial such that for each $x^K \in \mathrm{mon}(g)$, $\mathrm{coe}(x^K, \tilde{f}) = \frac{1}{K!}\mathrm{coe}(x^K, f)$. Then

$$\langle \tilde{f}, g \rangle = (f, g).$$

Claim A

There exist $(\beta_i)_{1\le i\le l} \in \mathbb{C}^l$ such that for $1 \le j \le l$,

$$\mathrm{coe}\left(x^{\tilde{K}_j}, \phi \sum_{i=1}^{l} \beta_i x^{K_i}\right) = \frac{1}{\tilde{K}_j!}\mathrm{coe}\left(x^{\tilde{K}_j}, \phi R(x)\right).$$

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Corollary

There exists $(\beta_i)_{1 \le i \le l} \in \mathbb{C}^l$ such that

$$\langle \phi \sum_{i=1}^{l} \beta_i x^{K_i}, \psi R(x) \rangle = (\phi R(x), \psi R(x)).$$

Assume $P = (p_1, \ldots, p_n), Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$.

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$$\begin{aligned}
(x^P, x^Q) &= (2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \prod_{j=1}^{n} e^{ip_j\theta_j} \overline{\prod_{j=1}^{n} e^{iq_j\theta_j}} d\theta_1 \ldots d\theta_n \\
&= \prod_{j=1}^{n} (2\pi)^{-1} \int_{\theta_j=0}^{2\pi} e^{i(p_j-q_j)\theta_j} d\theta_j \\
&= \begin{cases} 1, & \text{if } P = Q, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

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$$(f, g) = (2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \overline{g(e^{i\theta_1}, \ldots, e^{i\theta_n})} d\theta_1 \ldots d\theta_n.$$

Assume Claim A is true.

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By Lemma A,

$$0 = \langle \phi \sum_{i=1}^{l} \beta_i x^{K_i}, \psi R(x) \rangle = (\phi R(x), \psi R(x)).$$

$$
\begin{aligned}
& (\phi R(x), \psi R(x)) \\
= \; & (2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \phi\overline{\psi} |R(x)|^2 d\theta_1 \ldots d\theta_n \\
= \; & \pm(2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \prod_{1 \le i < j \le n} \left(-\frac{(e^{i\theta_i} - e^{i\theta_j})^2}{e^{i\theta_i} e^{i\theta_j}} \right)^{t_{ij}^+} \\
& \left(\frac{(e^{i\theta_i} + e^{i\theta_j})^2}{e^{i\theta_i} e^{i\theta_j}} \right)^{t_{ij}^-} |F(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 d\theta_1 \ldots d\theta_n \\
= \; & \pm(2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \prod_{1 \le i < j \le n} (2 - 2\cos(\theta_i - \theta_j))^{t_{ij}^+} \\
& (2 + 2\cos(\theta_i - \theta_j))^{t_{ij}^-} |F(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 d\theta_1 \ldots d\theta_n \\
\neq \; & 0,
\end{aligned}
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Let

$$A = (a_{ij})_{l \times l}$$

$$a_{ij} = (\phi x^{K_i}, \psi x^{K_j}) = (\phi x^{K_i}, x^{\tilde{K}_j}) = \mathrm{coe}(x^{\tilde{K}_j}, \phi x^{K_i}).$$

Let

$$b = (\frac{1}{\alpha_i \tilde{K}_i!} \mathrm{coe}(x^{\tilde{K}_i}, \phi R(x))_{1 \le i \le l}.$$

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Claim A says

$$\exists \beta \in \mathbb{C}^l, A\beta = b.$$

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Claim A says

$$\exists \beta \in \mathbb{C}^l, A\beta = b.$$

It suffices to show that A is non-singular.

We prove this by showing $\forall \alpha \in \mathbb{C}^{l}$,

$$\alpha A \alpha^{*} \neq 0.$$

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Let

$$F_\alpha(x) = \sum_{i=1}^{l} \alpha_i x^{K_i}.$$

$$\begin{aligned}\alpha A \alpha^* &= \sum_{1 \leq i,j \leq l} \alpha_i \overline{\alpha_j} (\phi(x) x^{K_i}, \psi(x) x^{K_j}) \\ &= (\phi F_\alpha(x), \psi F_\alpha(x)).\end{aligned}$$

$$
\begin{aligned}
& (\phi F_\alpha(x), \psi F_\alpha(x)) \\
= \; & (2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \prod_{1 \le i,j \le l} (e^{i\theta_i} - e^{i\theta_j})^{2t_{ij}^+} (e^{i\theta_i} + e^{i\theta_j})^{2t_{ij}^-} \\
& \overline{(e^{i\theta_i} e^{i\theta_j})^{t_{ij}^+ + t_{ij}^-}} |F_\alpha(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 d\theta_1 \ldots d\theta_n \\
= \; & (2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \prod_{1 \le i,j \le l} \left(\frac{(e^{i\theta_i} - e^{i\theta_j})^2}{e^{i\theta_i} e^{i\theta_j}} \right)^{t_{ij}^+} \left(\frac{(e^{i\theta_i} + e^{i\theta_j})^2}{e^{i\theta_i} e^{i\theta_j}} \right)^{t_{ij}^-} \\
& |F_\alpha(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 d\theta_1 \ldots d\theta_n \\
= \; & (-1)^{|T^+|} (2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \prod_{1 \le i < j \le l} (2 - 2\cos(\theta_i - \theta_j))^{t_{ij}^+} \\
& (2 + 2\cos(\theta_i - \theta_j))^{t_{ij}^-} \, |F_\alpha(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 d\theta_1 \ldots d\theta_n \\
\neq \; & 0.
\end{aligned}
$$


The End