Friendly Bisections of Random Graphs

Friendly & Unfriendly Partitions

Unfriendly Partⁿ: A partⁿ $A \cup B$ of $V(G)$ of a graph G s.t.

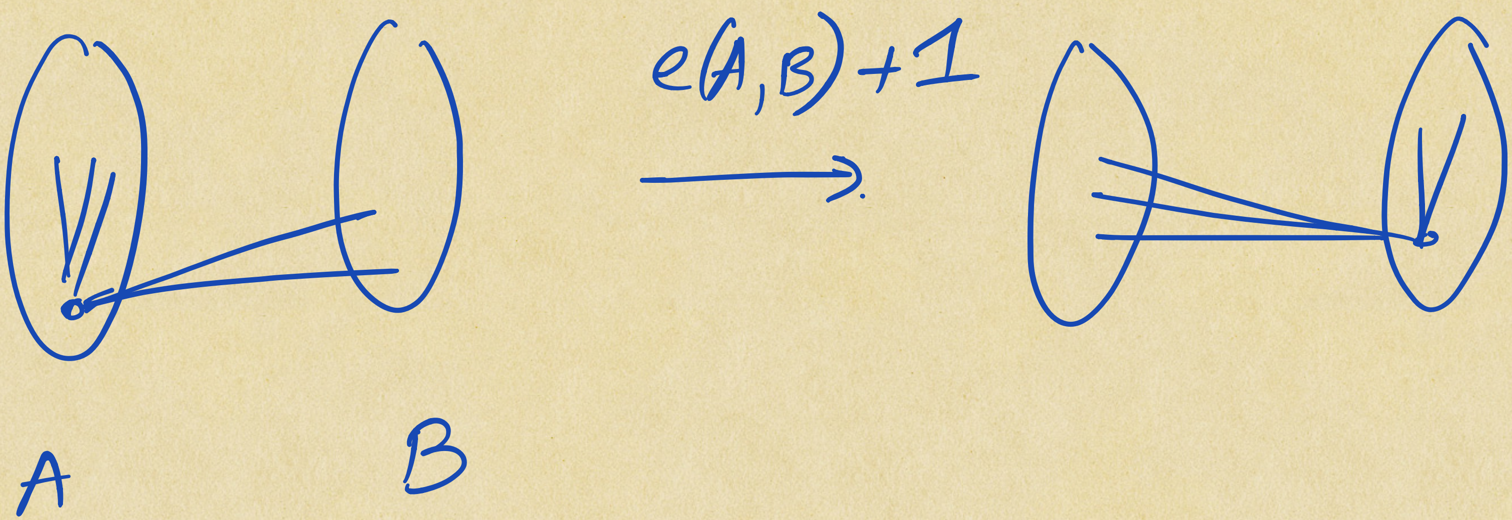
- $d_B(x) \geq d_A(x) \quad \forall\, x \in A$
- $d_A(x) \geq d_B(x) \quad \forall\, x \in B$

Friendly Partⁿ: Sim^r but more n'b'rs on same side.

Existence

Every graph has an unfriendly partition.

Eg: (A,B) which maximises $e(A,B)$ i.e. a max-cut

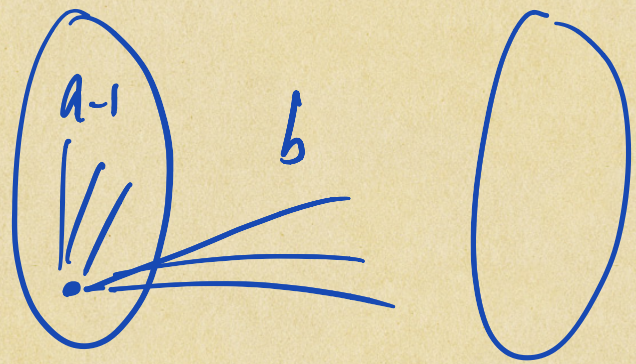


Existence

Friendly partitions need not exist.

Eg: $G = K_n$ $\quad n = a+b, \quad a \leq b$

this side is in trouble!



Set-Theoretic Aspect

Does every countable graph have an unfriendly partition? [Open]

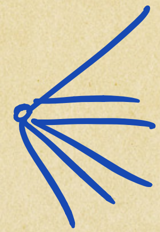
Shelah-Milner: No such part^n for bigger cardinalities.

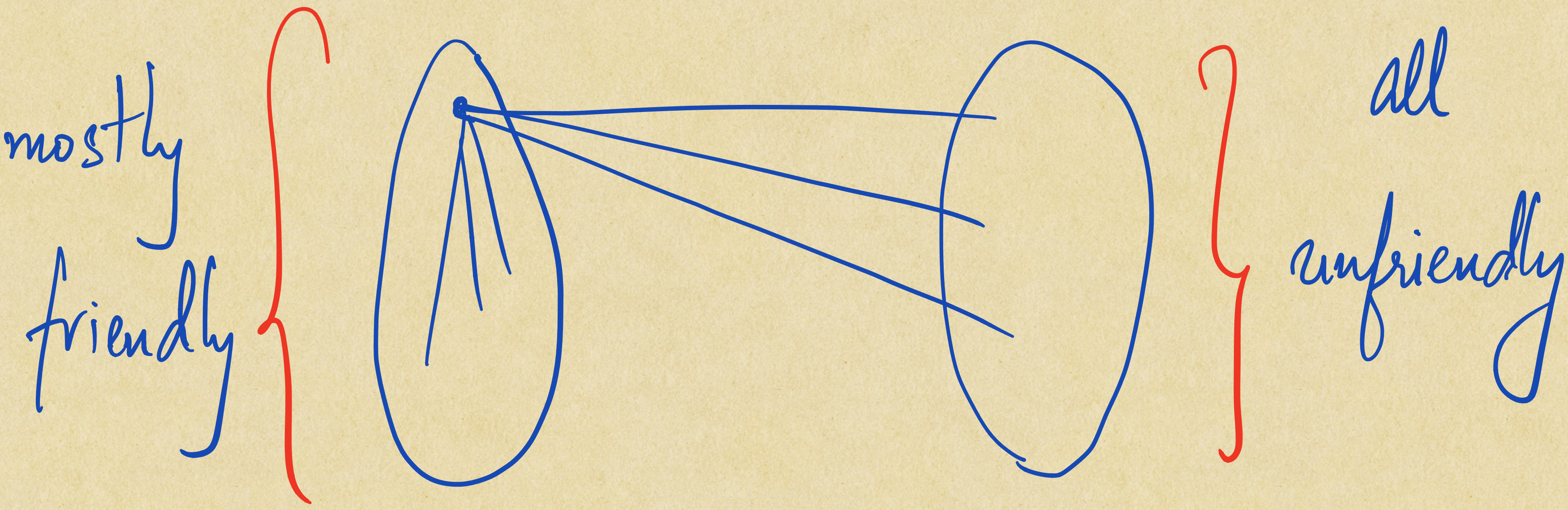
Bisections

What if we want a friendly or unfriendly **bisection**?

$V(G) = A \cup B$ w/ $|A| = |B|$.

(Unsurprisingly) Need Not Exist

G = a star i.e. 



What about <u>typical</u> graphs?

Exceptions (stars, cliques) are rare.

<u>Conjecture</u> [Füredi, 88] $G \sim G(n, 1/2)$ whp has a bisection where $n - o(n)$ vertices friendly.

Our Results

Theorem (Ferber, Kwan, N, Sah, Sawhney).

Füredi's conjecture is true.

[1st result for unfriendly bisections.]

"Almost" Bisections are Easy

Stiebitz: Suppose $d(x) \geq a(x) + b(x) + 1 \ \forall x \in V(G)$.

$\exists$ a part^n $A \cup B$ of $V(G)$ s.t.

- $d_A(x) \geq a(x) \quad \forall x \in A$
- $d_B(x) \geq b(x) \quad \forall x \in B$

Apply Stiebitz to $G(n, 1/2)$ w/ $a(x) = b(x) = \left\lfloor \frac{d(x)-1}{2} \right\rfloor$

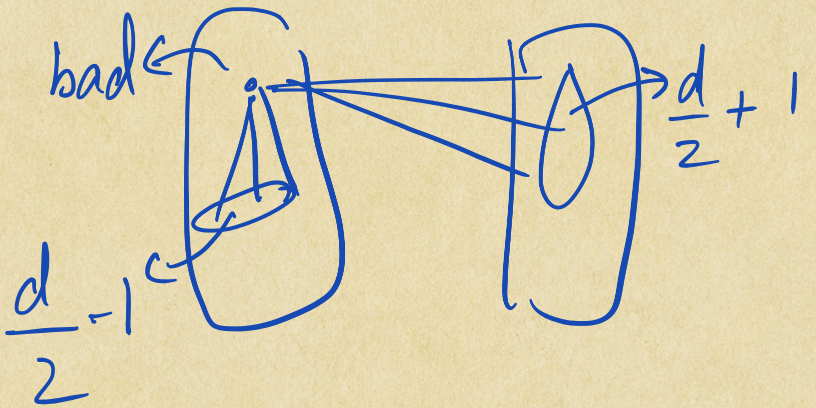
Get a partⁿ $A \cup B$ w/ $d_A(x) \geq \left\lfloor \frac{d(x)-1}{2} \right\rfloor$

$\forall x \in A$ & lll^{y} for B.

Two things: ① $n - o(n)$ v^x actually friendly

② $|A| \sim |B|$ (in fact $\approx \sqrt{n}$ apart)

(1) Most vertices are friendly: Fix $A \cup B$ & reveal $G[A]$



$\mathbb{P}[\text{bad}] \sim 1/\sqrt{n}$ &

indep. $\forall v \in A$.

$$\Rightarrow \mathbb{P}\left[\geq \frac{10n}{\log n} \text{ bad in } A \text{ or } B\right] \lesssim \left(\frac{1}{\sqrt{n}}\right)^{10n/\log n} = o(2^{-n})$$

Done by union bound over all part^n.

(2) $|||^y$ by union bound.

Actual Bisections - Idea.

Swapping Algorithm

① Start w/ any bisection $A \cup B$

② Pick an unfriendly pair $x \in A$, $y \in B$

& swap x & y.

③ Stop when stuck.

Obstacles to Analysis

This works in practice (i.e. experimentally)

<u>but</u> hard to analyse: Why?

<u>Correlations</u>! After $n/100$ steps, we've

lost a lot of independence.

What We Do

① Swap in chunks i.e. two sets of εn unfriendly vertices from A & B.

[like Rödl nibble – $O_\varepsilon(1)$ swaps suffice.]

② Swap another random (smaller) pair of $\varepsilon^{10} n$ vertices [reduces correlation]

Ingredients in Analysis

Define $\Delta(v)$, given $A \cup B$, to be

$d_A(v) - d_B(v)$ if $v \in A$ (& $|||^{ly}$ for B).

Set: $\Delta(A, B) = \sum_v \Delta(v)$.

Increment Lemma: If a swap is possible

$$\Delta' = \Delta + \Omega(n^{3/2}).$$

Aux: Concentration

Concentration Lemma: Repeat swaps $k = O(1)$ times. Let X_k & Y_k be the # of unfriendly vertices in A_k & B_k at this stage.

The, whp, $|X - Y| = o(n)$.

Lemmas $\Rightarrow$ Theorem

By increment, # swaps is $O(1)$.

If we cannot swap, either A or B has $\leq \varepsilon n$ unfriendly vertices, but by conc^{n}, both parts have $\leq \varepsilon n$ vertices.

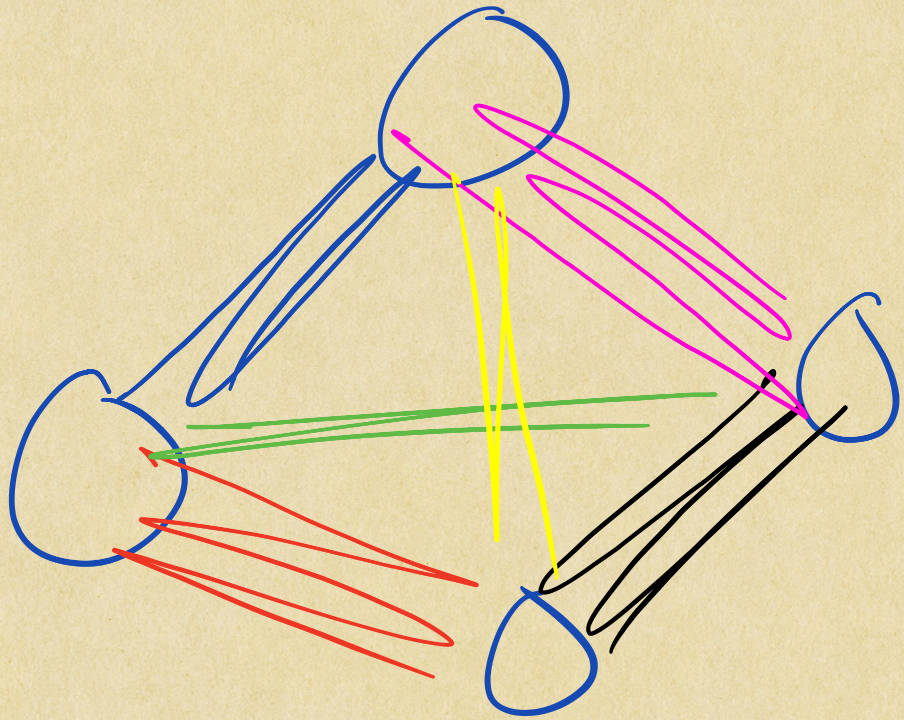
How to <u>prove</u> concentration?

Many ingredients, but roughly:

- at stage k, there are $\underline{2^k}$ types of vertices (based on history).
- show all such sets have size what we expect.

McKay-Wormald

To track further:



$G[\text{Part}_i, \text{Part}_j]$

$\hookrightarrow G(n, 1/2)$ w/ degree information revealed $\longrightarrow$ McKay-Wormald.

Open Problems

Conjecture 1 : Whp, any min-bisection of $G(n, 1/2)$ is almost-friendly

Conjecture 2 : Whp, $G(n, 1/2)$ has a (fully) friendly bisection.