Vertex Sparsification for Edge-Connectivity

Bundit Laekhanukit

ITCS@SUFE

Collaborators (8 Authors)

- Parinya Chalermsook (Aalto)
- Syamantak Das (IIIT Delhi)
- Junbum Kook (KAIST)
- Yang P. Liu (Stanford)
- Richard Peng (Georgia Tech)
- Mark Sellke (Stanford)
- Daniel Vaz (TU Munich)

Graph Sparsifiation & Compression

Question:

Can we keep a graph with smaller size

while preserving some "important" property?

Complete Graph vs Expanders

- Dense vs Sparse
- Preserve (approximately)
  - Connectivity
  - Distance
  - Vertex Expansion (or throughput)

Gomory - Hu Tree [GH'61]

- Flow sparsifier

  - For every graph $G = (V, E)$

    $\exists$ a weighted tree $T = (V, E')$ s.t.

    $$\text{maxflow}_G(s,t) = \text{maxflow}_T(s,t)$$

    for all nodes $s,t \in V$

Cut-Preserving Sparsifiers - Mimicking Network

- Given graph $G=(V,E)$, k terminals $T \subseteq V$,

  we want a graph $H=(V',E')$, $T \subseteq V'$ s.t.

  $$\text{mincut}_G(A,B) = \text{mincut}_H(A,B)$$

  for every partition $A \uplus B = T$

  - H may not be a subgraph of G,
    but we want some vertices of H to corresp. to T.

Existence of Mimicking Network

- Every graph G with k terminals has a mimicking network of size $O(2^{2^k})$

[Hagerup, Katajainen, Nishimur and Ragde '95]

[Khan, Ragavendra, Tetali and Végh 2014]

Impossibility

$\exists$ a graph G on k terminals s.t.

every mimick network has size

$$\Omega(2^k)$$

Barrier

[Krauthgamer & Rika 2013]

Algorithmic Applications

- Meta algorithm for approximation algorithms

  [Moitra 2009, Chuzhoy 2012]

our results
- Speed-up Dynamic Program
- Data-Structure for dynamic queries

Can we break 2^k barrier

if we only need to preserve cut

of values $\leq c$?

In most applications, c can be constant, say 2, 3, 4, 5.

Threshold Cuts

$$\boxed{\mathrm{mincut}^{c}_{G}(A,B) = \min\{\mathrm{mincut}_{G}(A,B),\ c\}}$$

⇒ c-mimicking network H:

$$\mathrm{mincut}^{c}_{G}(A,B) = \mathrm{mincut}^{c}_{T}(A,B)$$

for all $A \uplus B = T$

Our Results

poly on k

Existence Result $\Rightarrow$ • c-mimicking network H of size $O(kc^6)$ exists

and $\boxed{H \text{ is a minor of } G}$ can be obtained of contraction

skip

• Optimal-sized contraction-based mimicking network

can be computed in time $O(m(c \log n)^{O(c)})$

poly (k)

• c-mimicking network of size $k \cdot O(c)^{2c}$ can be

computed in time $m \cdot c^{O(c)} \cdot \text{polylog}(n)$

Applications

- Finding min-cost k-connected subgraph in time $tw(G)^{O(tw(G)\,\mathrm{poly}(c))}$ ← improved from $2^{2^{2^{2^{tw(G)}}}}$

- Offline dynamic c-edge connectivity in time $\tilde{O}(c^{O(c)})$ per queries

(improved from $O(\sqrt{n})$ [Molina-Sandlund'18]

Contraction & Properties

- Obs : Contraction never decreases

  Edge - Connectivity

  But, it may INCREASE edge-connectivity

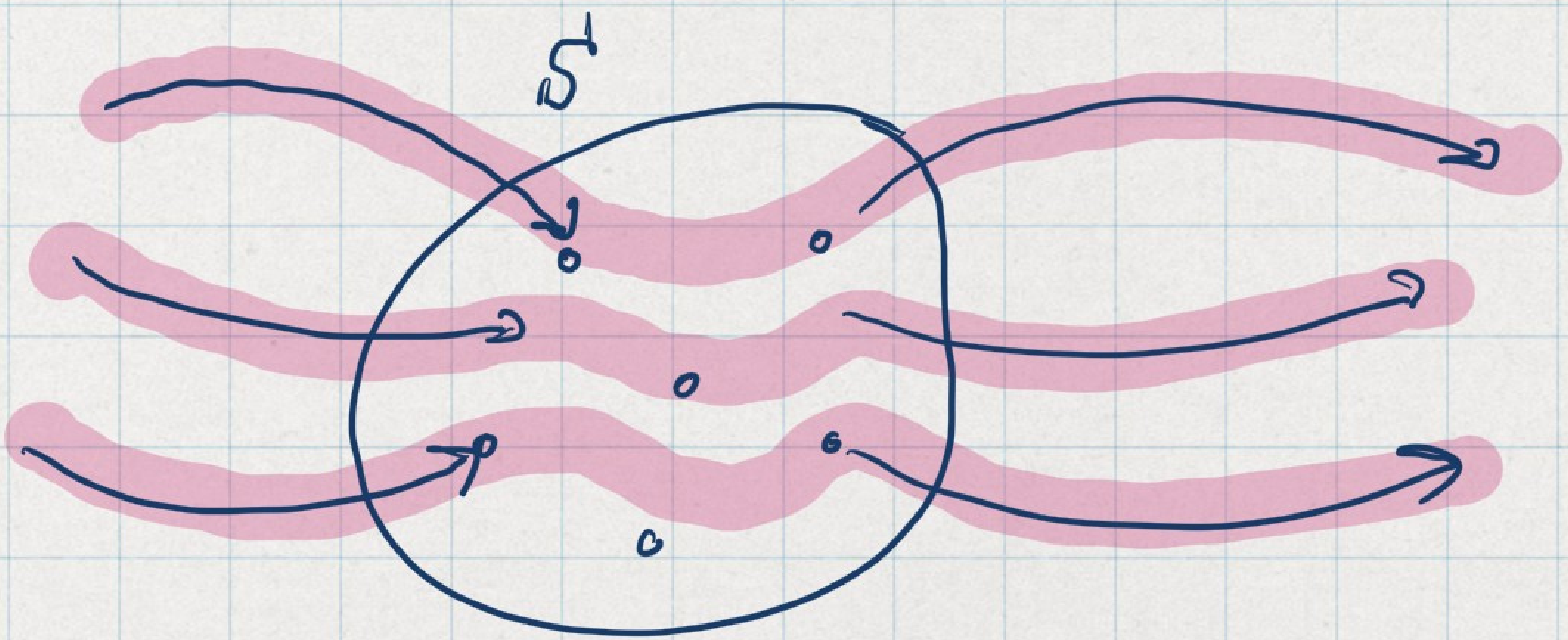
- Key Idea

Contract an edge as long as it does not increase edge-connectivity of any terminal-cut (A, B)

$\Rightarrow$ $\boxed{\text{How can we determine which sequence of contraction is Good ?}}$

results in c-mimicking net with small-size

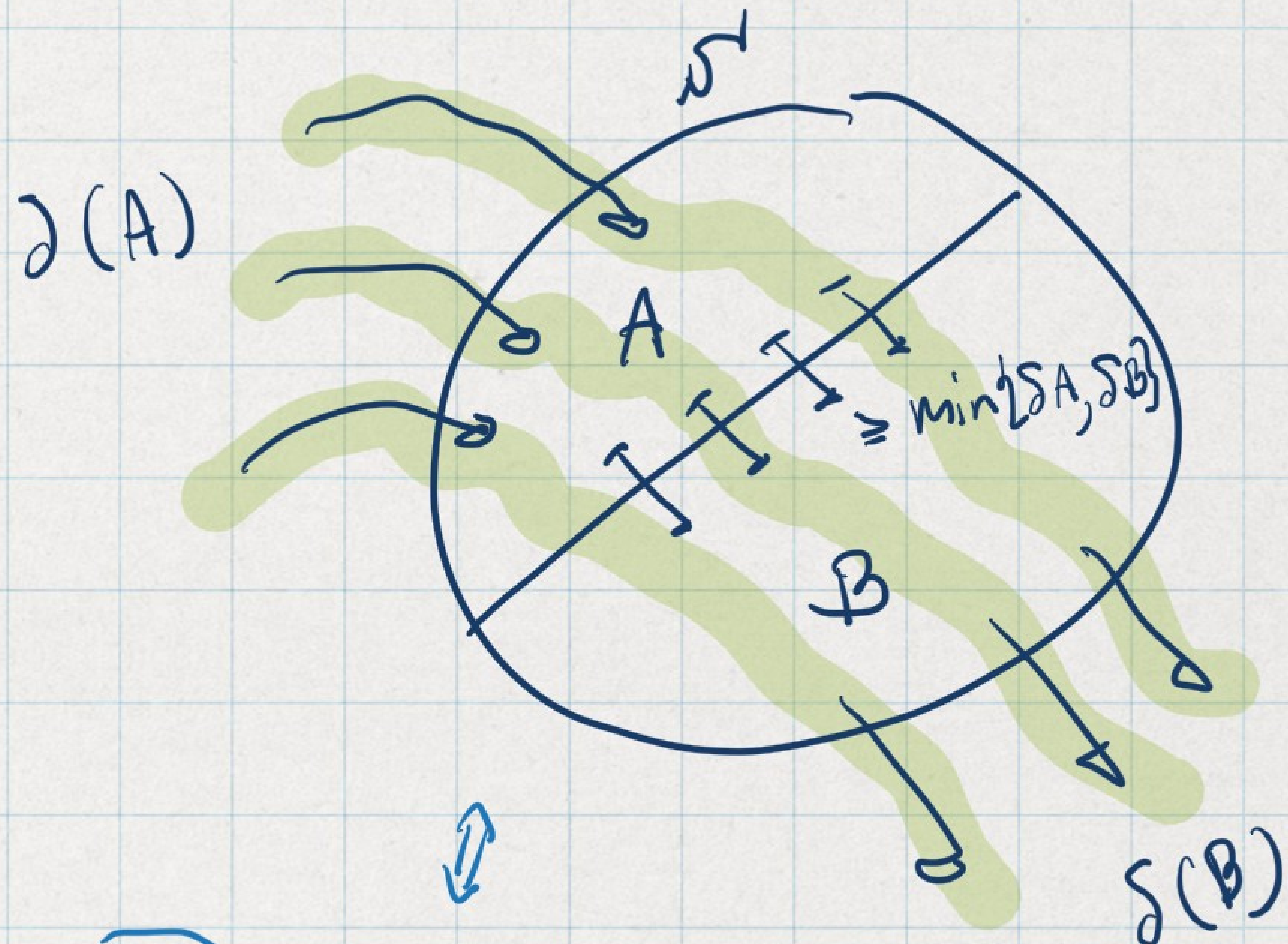
Good Set to contract - Well-linked Set

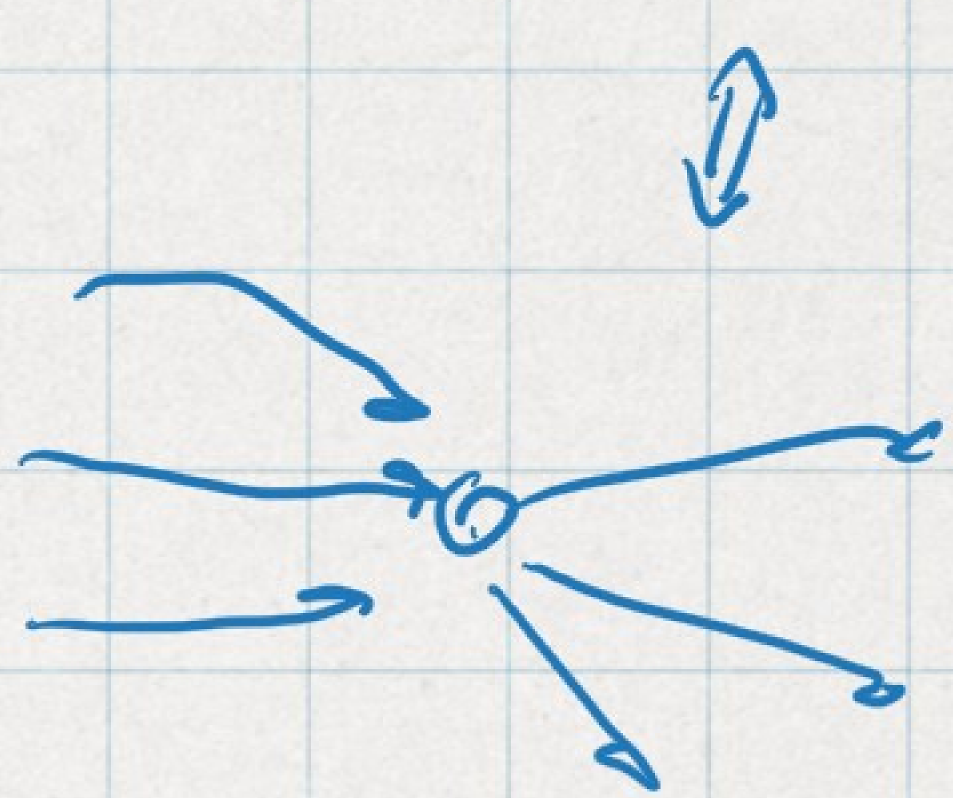


Possible notions
- $G[S]$ is at least c-edge-connected.

Highly likely that it does not exist.

(Ref Define Well-linkedness of a set S.





S is well-linked if

$\forall\, A \uplus B = S$,

$E_G(A,B) \geq \min\{\delta A, \delta B\}$

threshold c-well linked if

$\forall\, A \uplus B = S$,

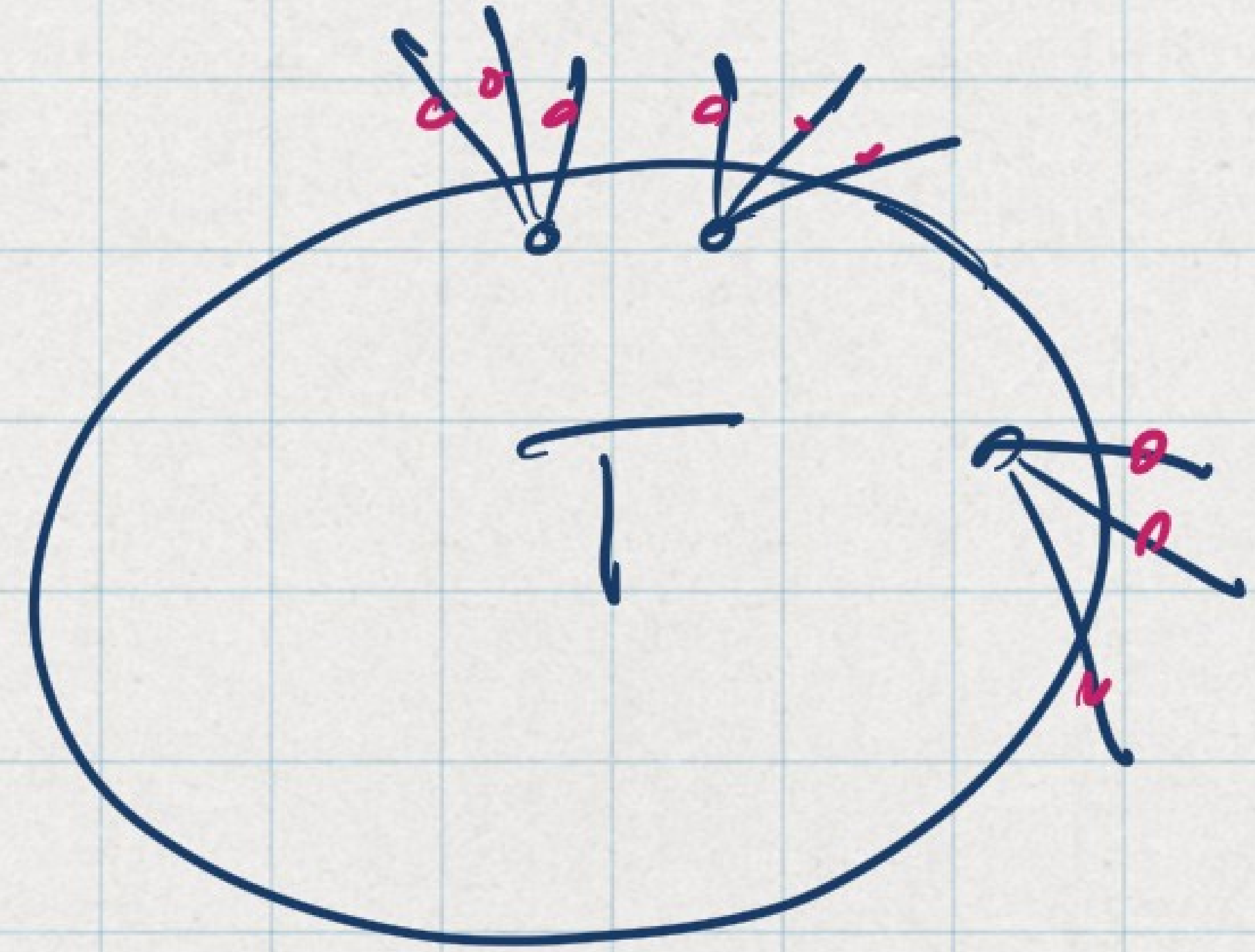
$E_G(A,B) \geq \min\{\delta A, \delta B, \underline{c}\}$

Contracting c-well linked set

does not increase threshold

c-Edge connectivity

Transform terminal set



Pull out the terminal to each edge

$\Rightarrow$ $|T'| = O(k \cdot c)$

- Assume WLOG, $t \in T$ has c edges outside T.
  $|E_G(t, V-T)| = c$

Construction Overview $\Rightarrow$ c-Mimicking Network of size kc^4

- Find c-well like set $\Rightarrow$ How to find it?

- Partition $V \cap T = S_1 \uplus \dots \uplus S_p$

  - If S_i is c-well-linked, we are done...

  Otherwise, we recursively refine S_i

Notion of Progress ⇒ Potential Functions

$$\boxed{\Phi(\hat{S}) = \sum_{i=1}^{p} \partial S_i}$$

init $\Phi(\hat{S}) \leq kc$

After $\partial S_i \leq 2c-1$ for all i
or S_i is c-well-linked.

need to handle

$\boxed{\circledast \quad \partial S_i \geq 2c \text{ and } S_i \text{ is not } c\text{-well-linked.}}$

Violating cut

$\boxed{S_i : S_j \text{ is not } c\text{-well-linked and } \delta S_j \geq 2c \Rightarrow \exists (A,B) \text{ that violates } c\text{-well link.}}$

$\Rightarrow$ We remove S_j add A, B to our partition

$A \uplus B = S_j$ is the partition of S_j

that violates c-well linkedness.

Finally,

- $\boxed{S_j \text{ s.t. } S_j \text{ is } c\text{-well-linked.}}$ Bounded by kc
- S_j s.t. $\partial S_j \leq 2c-1$

By matroid theory, $\exists$ a representative set of size $O(c^5)$ [Krastsch-Wahlström'12]

Counting # pieces at the end.

$$O(kc \cdot c^5) = \boxed{O(kc^6)}$$

#

Conclusion and Remark

- $\exists$ a c-mimick network of size $O(kc^6)$

  $\Rightarrow$ This was improved by Yang P. Liu to $O(kc^3)$ in $n^{O(1)}$ time.

- A fast algorithm is done by exploiting nearly linear time Expander Decomposition [Saranurak–Wang'19] to partition $V-T$