On coloring of graphs of girth $2\ell+1$ without longer odd hole

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Joint work with Di Wu & Yian Xu

- Basic definitions and background
- Main results
- Some proofs

Definitions & background

Graph $G = (V(G), E(G))$, $S \subseteq V(G)$.

$G[S]$: the subgraph induced by S.

Clique: S is called a clique if $G[S]$ is complete.

stable set: S is called a stable set if $G[S]$ has no edges.

Clique number: $\omega(G) = \max\{|S| : S \text{ is a clique of } G\}$.

stable number: $\alpha(G) = \max\{|S| : S \text{ is a stable set of } G\}$.

Let k be a positive integer.

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$$\boxed{\chi(G) \geq \omega(G) \text{ for all graphs } G.}$$

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χ-bounded problem: (Gyárfás, 1975) Let $\mathcal{G}$ be a family of graphs. If there exists a function f s.t. $\chi(G) \le f(\omega(G))$ for each $G \in \mathcal{G}$, then we say that $\mathcal{G}$ is χ-bounded, and called f a binding function of $\mathcal{G}$.

$\mathcal{F}$-free graph: Let $\mathcal{F}$ be a set of graphs. We say that a graph G is $\mathcal{F}$-free if G induces no member of $\mathcal{F}$ as its subgraph.

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Even/odd hole: An even (resp. odd) hole is a hole of even (resp. odd) length.

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$\boxed{\text{The } \chi\text{-bounded problem of some holes-free graphs are studied extensively}}$

Erdős proved that for any positive integer k and l, there exists a graph G with $\chi(G) \geq k$ and without cycles of length less than l. (1959)

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Let I be a set of positive integers, and let

$$\mathcal{C}_I = \{\text{cycle of length } i,\ i \in I\}.$$

To guarantee the χ-boundedness of $\mathcal{C}_I$-free graphs,

$|I|$ cannot be finite !!!

Hole free graphs are also called chordal graphs, which have $\chi(G) = \omega(G)$.

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Perfect graph: A graph G is said to be perfect if $\chi(H)=\omega(H)$ for each induced subgraph H of G.

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Theorem (The Strong Perfect Graph Theorem, Chudnovsky et al, 2006)

A graph is perfect if and only if it induces neither odd holes nor their complements.

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Theorem (Addario-Berry, Chudnovsky, Havet, Reed, and Seymour 2008, and Chudnovsky and Seymour 2020+)

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☆ One may find other results on even hole free graphs in a survey of Vušković

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Confirming three conjectures of Gyárfás,

Scott and Seymour (2016) proved that $\chi(G) \le \frac{2^{2^{\omega(G)+2}}}{48(\omega(G)+2)}$ for every odd hole free graph G,

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☆ One may find more results and questions in a survey of Scott and Seymour (2020), and a survey of Schiermeyer and Randerath (2019).

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In 2014, Plummer and Zha presented some counterexamples to Robertson's conjecture, and proposed the following questions.

Question 1: How close are graphs of $\mathcal{G}_2$ to perfect graphs?

Question 2: Are the graphs of $\mathcal{G}_2$ have bounded chromatic number?

Question 3: Is it true that $\chi(G) \leq 3$ if $G \in \mathcal{G}_2$?

Answering Questions 1 and 2, Xu, Yu, and Zha proved

Theorem (X., Yu, and Zha, 2017)
Let G be a graph of $\mathcal{G}_2$, and let u be a vertex of G. Then, for every positive integer h, the set of vertices of distance h to u induces a bipartite subgraph.

Answering Questions 1 and 2, Xu, Yu, and Zha proved

Theorem (X., Yu, and Zha, 2017)
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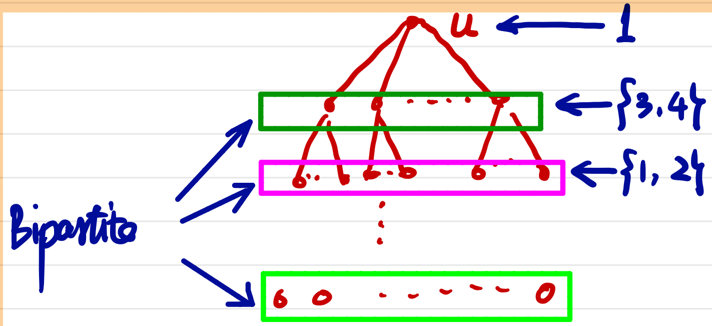
As a consequence,

$\chi(G) \leq 4$ if $G \in \mathcal{G}_2$.



Main results

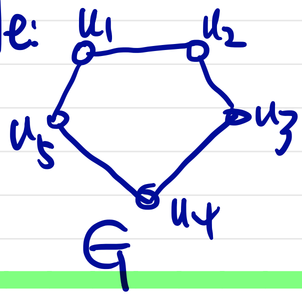
Let S be a subset of $V(G)$, and $i \geq 0$.

For $x \in V(G)$, we define

$$d(x, S) = \min\{d(x, u) : u \in S\},$$

$$L_i(S) = \{v : d(v, S) = i\}.$$

Example:



$S = \{u_1, u_2\}$. $L_0(S) = S$

$L_1(S) = \{u_3, u_5\}$, $L_2(S) = \{u_4\}$

$L_i(S) = \phi$ if $i \geq 3$

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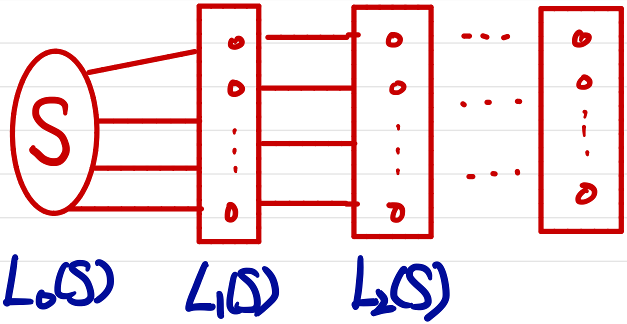

If $j > i+1$, then $uv \notin E(G)$ for $u \in L_i(S)$ and $v \in L_j(S)$.

Theorem 1 (Wu, X. and Xu, 2021+)

Let $l \geq 2$, $G \in \mathcal{G}_l$, and $S \subseteq V(G)$ s.t. $G[S]$ is connected. If, for each $1 \leq i \leq l-1$, $L_i(S)$ induces a bipartite subgraph, then $G[L_i(S)]$ is bipartite for each $i \geq 1$.

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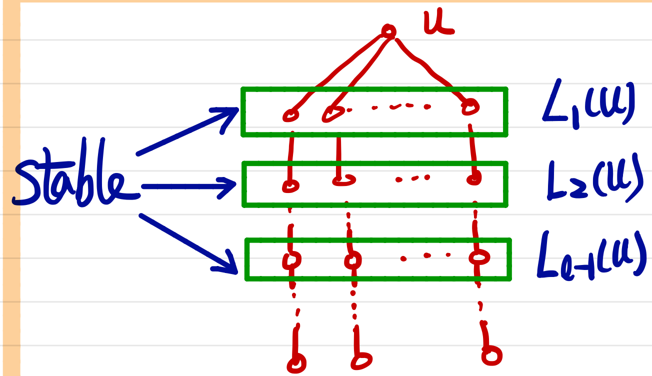
By taking $S = \{u\}$, $L_i(u)$ is stable if $1 \leq i \leq \ell-1$, and so $L_i(u)$ is bipartite for all i.



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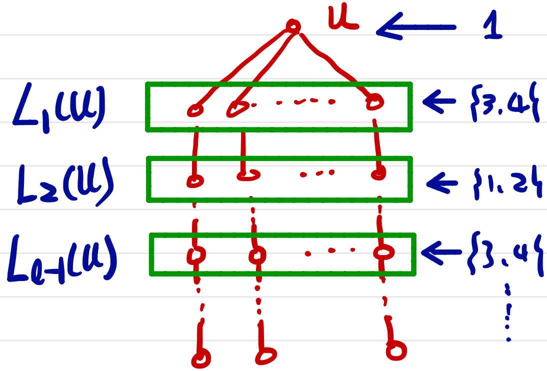

$\chi(G) \le 4$ if $G \in \bigcup_{l \ge 2} \mathcal{G}_l$.

Theorem 2 (Wu, X., and Xu, 2020+)

Let $G \in \mathcal{G}_5$. If G induces no two 5-cycles sharing edges, then $\chi(G) \leq 3$.

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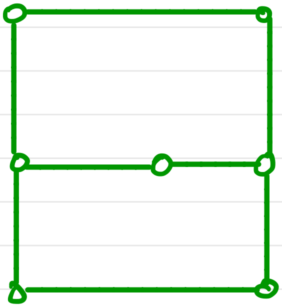
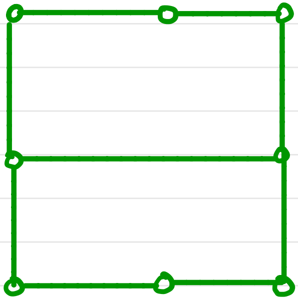
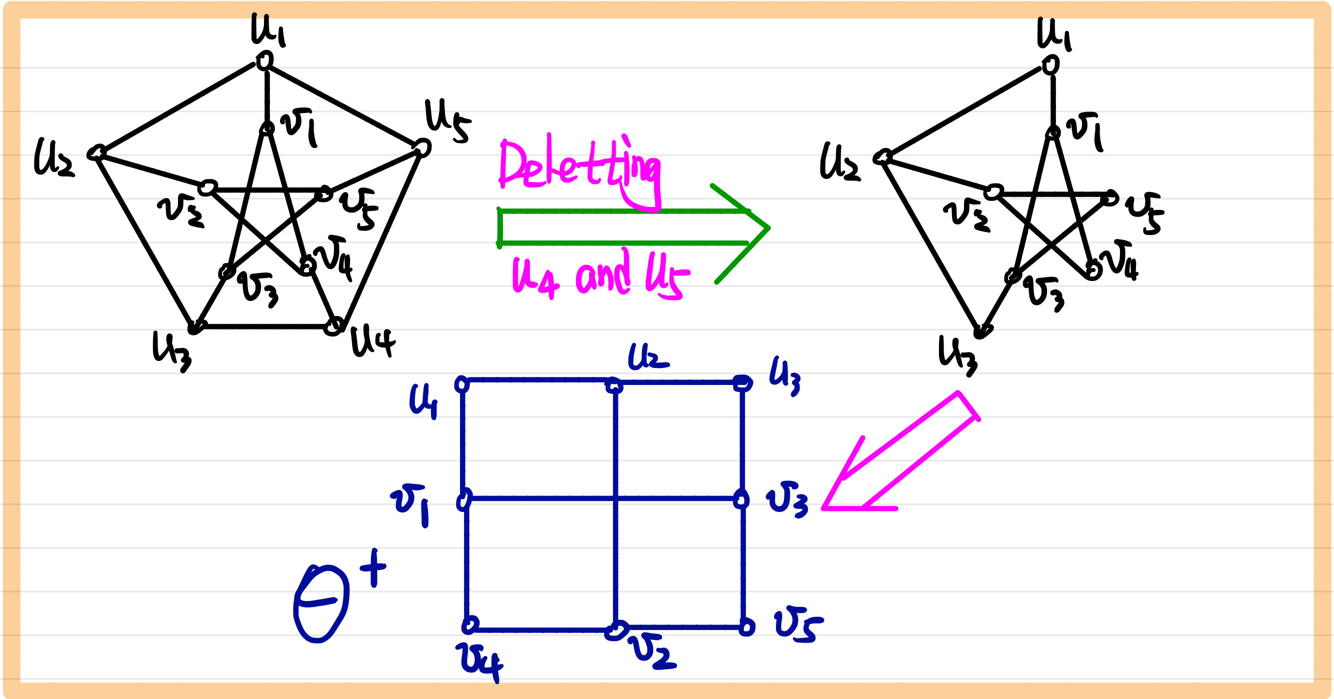
Two 5-cycles sharing edges in graphs of $\mathcal{G}_2$.



θ^-



θ


u1
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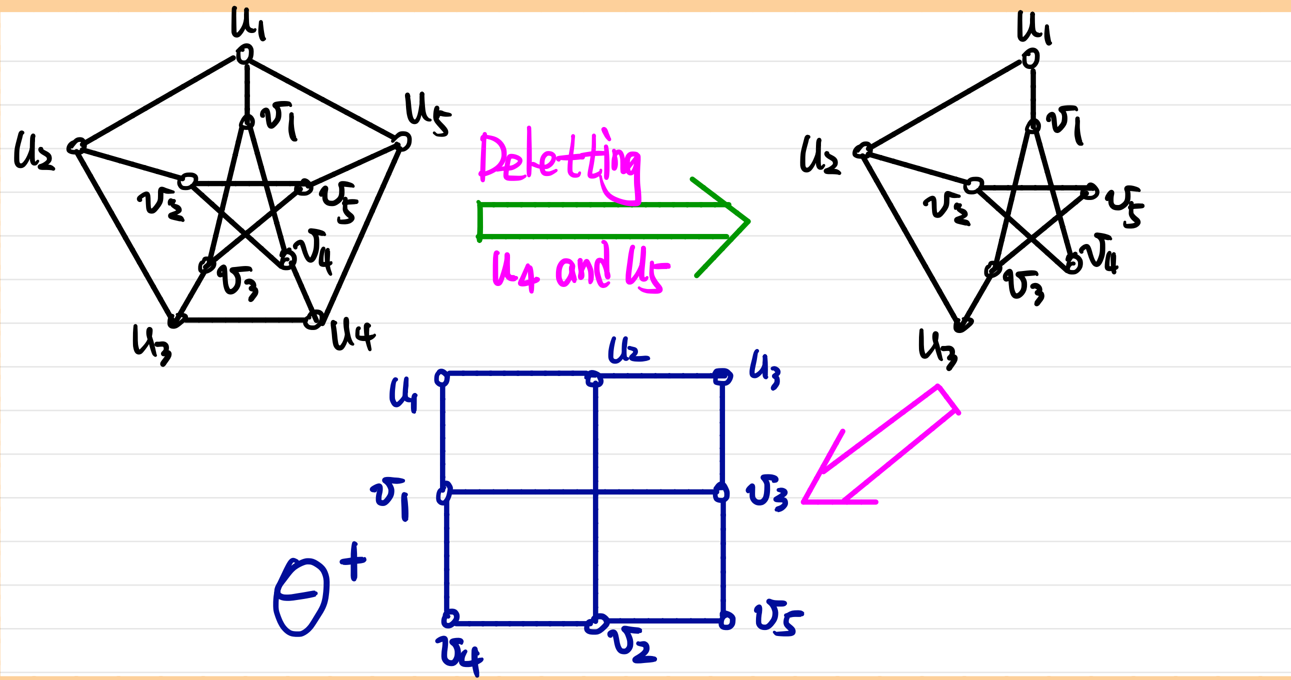
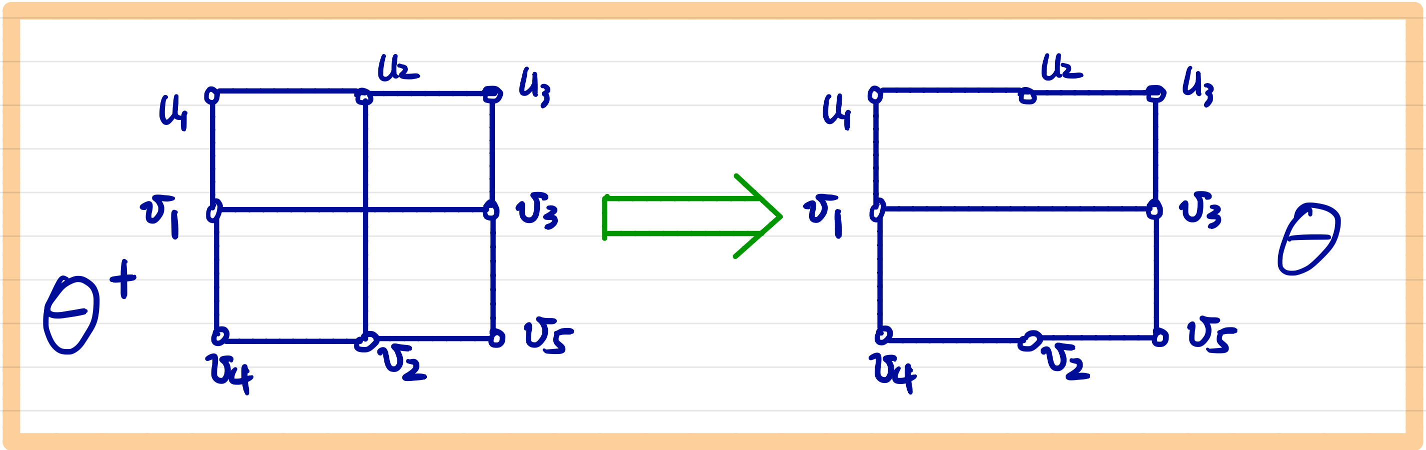
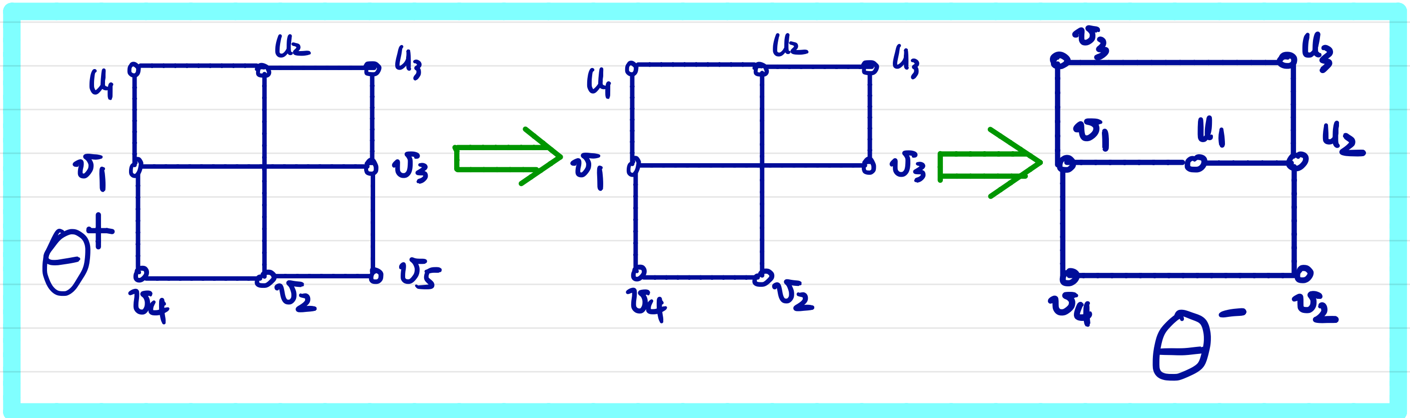


Theorem 3 (Wu, X., and Xu, 2020+)
Let $G \in \mathcal{G}_2$. If $\chi(G)=4$ but $\chi(H)\le 3$ for each proper subgraph H of G, then G is Θ^+-free.

Relation between Θ, Θ^-, and Θ^+





Sketches of some proofs.

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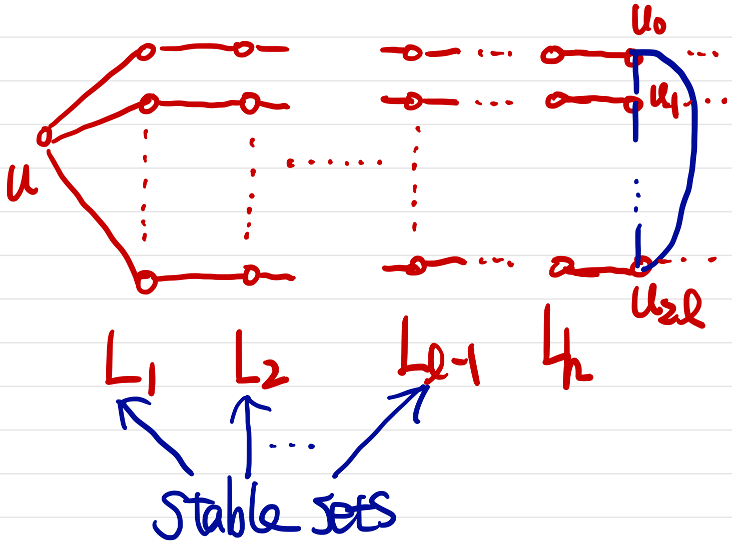
If, for each $1 \leq i \leq \ell - 1$, $L_i(S)$ induces a bipartite subgraph, then $G[L_i(S)]$ is bipartite for each $i \geq 1$.

We take $S = \{u\}$ for a single vertex u as example to show the procedure of proving Theorem 1.

Let $L_i = L_i(u)$, $i \geq 0$. Then, L_i, $0 \leq i \leq l-1$, is stable.

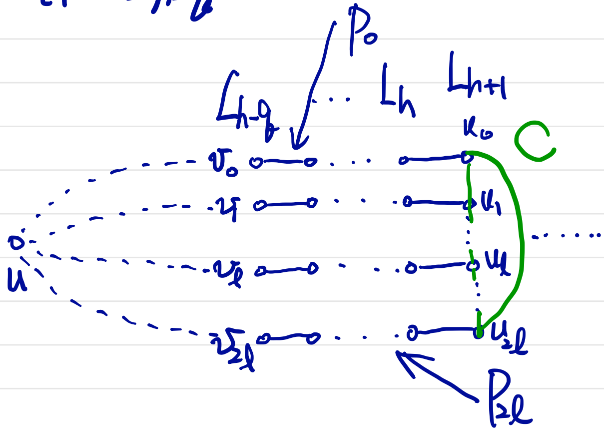
Let h be the smallest integer s.t. $G[L_{h+1}]$ is not bipartite, and let $C = u_0 u_1 \cdots u_{2l} u_0$ be an odd hole of $G[L_{h+1}]$.



Let $q = \lfloor \frac{\ell}{2} \rfloor$.

$\{v_0, v_1, \dots, v_{2\ell}\} \subseteq L_{h-q}$

For $i \in \{0, 1, \dots, 2\ell\}$, P_i be a $v_i u_i$-path of length $q+1$, s.t. $|\{v_0, v_1, \dots, v_{2\ell}\}|$ is minimum.



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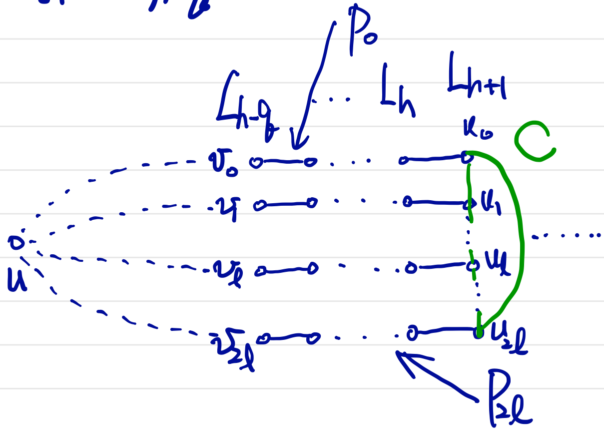
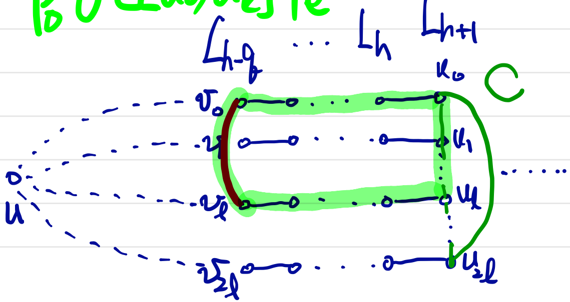
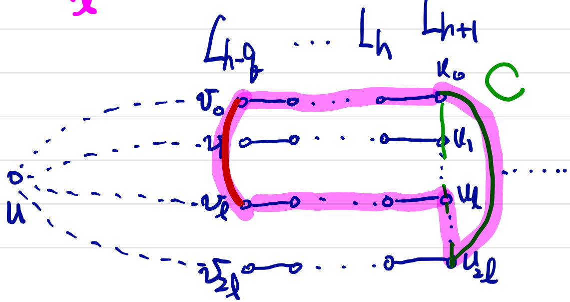


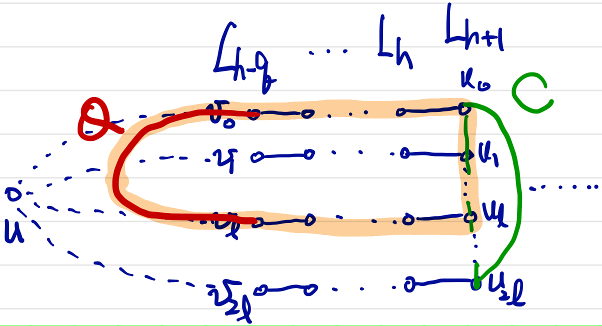
If $v_0 \sim v_\ell$, we have an odd hole of length at least $2\ell+3$,
$P_0 \cup C[u_0, u_\ell] P_\ell \cup v_0 v_\ell$ or $P_\ell \cup C[u_\ell, u_0] P_0 \cup v_0 v_\ell$

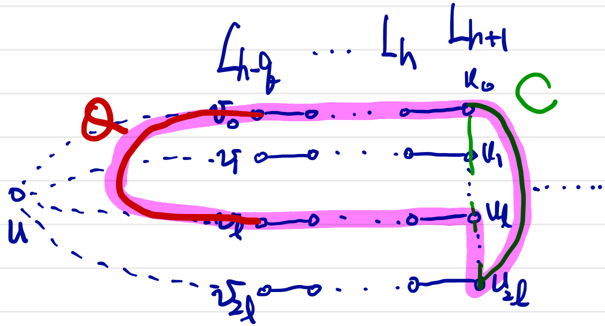




Suppose that $v_0 v_\ell \notin E(G)$.

If $v_\ell \neq v_0$, let Q be a shortest $v_0 v_\ell$-path with internal in $\bigcup_{i=0}^{h-g-1} L_i$, then we have an odd hole $\geq 2\ell+3$ in $Q \cup P_0 \cup P_\ell \cup C$.





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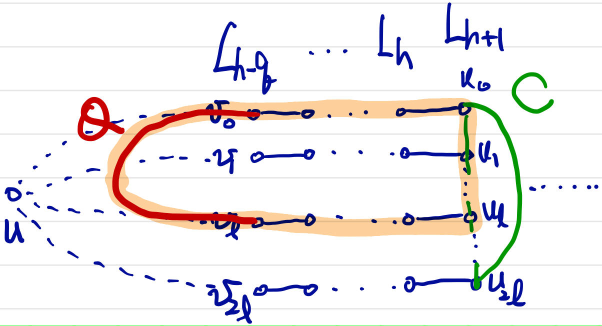
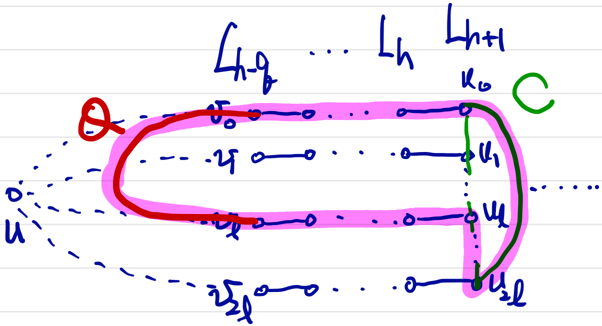




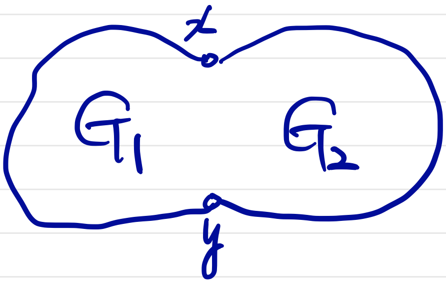
Now, we have $v_0 = v_1 = \cdots = v_{2\ell}$ by symmetry, and thus a hole, $v_0 P_0 u_0 v_1 P_1 v_0$, of length $\leq 2\ell$ occurs if $\ell > 2$, or a 7-hole $v_0 P_2 u_2 u_3 u_4 u_0 P_0 v_0$ occurs if $\ell = 2$. //

To prove Theorem 2, we need some properties of minimal non-3-colorable graphs in $\mathcal{G}_2$.

Lemma 1 (Wu, Xi, and Xu, 2020+)
Let G be a minimal non-3-colorable graph of $\mathcal{G}_2$, u a vertex and $\{u_1, u_2, u_3\} \subseteq N(u)$.
Then, G is 3-connected, every 3-cutset is stable, and $\{u, u_1, u_2, u_3\}$ is not a cutset.

We only prove its 3-connectivity.

Suppose that $\{x, y\}$ is a 2-cut.



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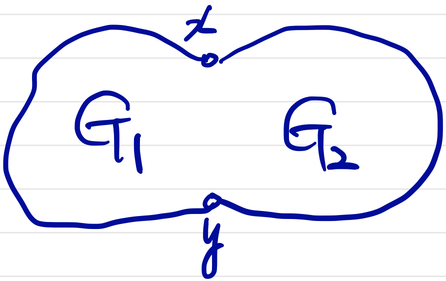
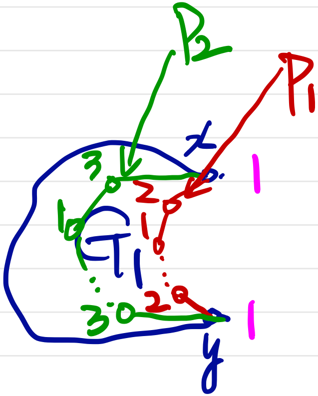
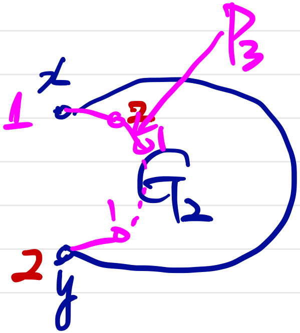






Both P_1 and P_2 have even length, one has length ≥ 4.

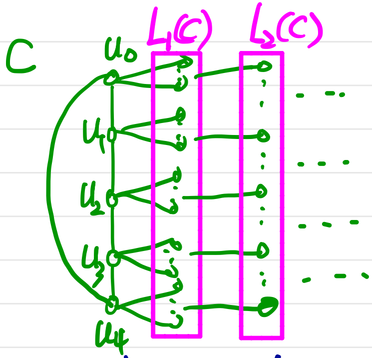
P_3 has odd length ≥ 3.

Now, an odd hole ≥ 7.

Then, Theorem 2 is a consequence of the following Lemma.

Lemma 2 (Wu, X., and Xu)
Let G be a 3-connected graph in S_2 that has a 5-hole.
If G induces neither Θ nor Θ^-, then G has a cutset $\{x, y, z\}$ with $xy \in E(G)$.

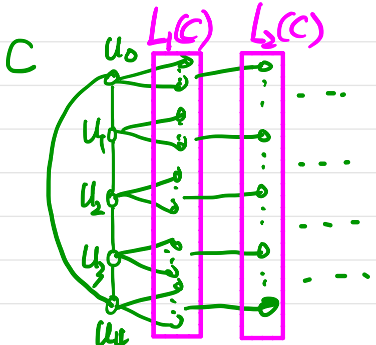
Θ^- Θ



A counterexample.

1) $L_1(C)$ is stable,

2) $N(u) \cap N(v) \cap L_2(C) = \phi$ for distinct u and v in $L_1(C)$.



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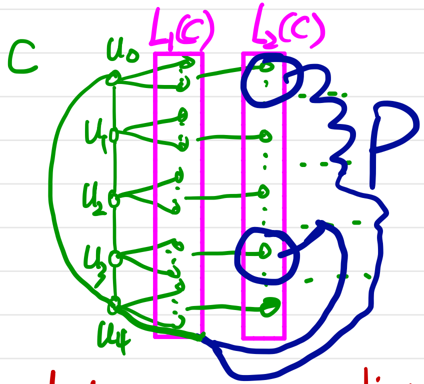
A counterexample.

The key is that if $G-C$ has an induced path P from $L_2(C) \cap L_2(u_i)$ to $L_2(C) \cap L_2(u_{i+3})$, then

$P \cap (N(u_{i+1}) \cup N(u_{i+2})) = \phi$ and

$P \cap N(u_{i+4}) \neq \phi$.



This is true for each i. We can deduce a contradiction.

Theorem 2 (Wu, X., and Xu, 2020+)
Let $G \in \mathcal{G}_3$. If G induces neither Θ nor Θ^- then $\chi(G) \leq 3$.

Suppose to its contrary, we choose G to be a minimal counterexample. Then, G has 5-hole as otherwise G would be bipartite. By Lemma 1, G is 3-connected and every 3-cutset is stable. But this contradicts the conclusion of Lemma 2. //

Theorem 3 (Wu, X., and Xu, 2020+)
Let $G \in \mathcal{S}_2$. If $\chi(G) = 4$ but $\chi(H) \le 3$ for each proper subgraph H of G, then G is θ^+-free.

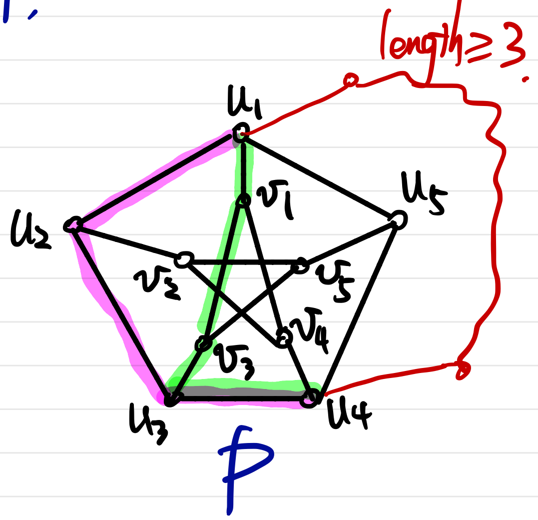
Theorem 3 is a consequence of Lemma 1 and the following

Lemma 4: (Wu, X, and Xu, 2021+)
Let G be a 3-connected graph in $\mathcal{S}_2$. If G has no unstable 3-cutset and is not the Petersen graph, then G is θ^+-free.

sketch of proof:

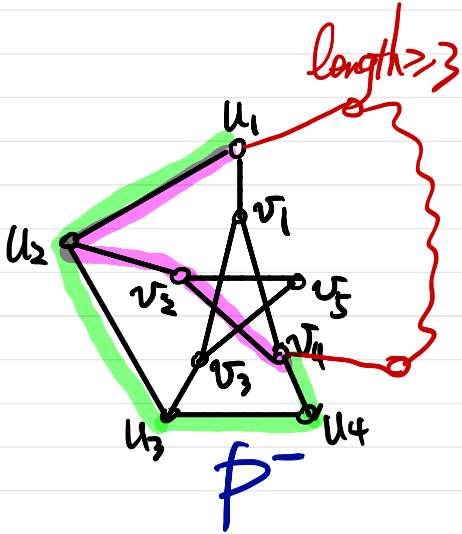
We first show that G does not induce the Petersen graph.

If G induces the Petersen graph P,
then every vertex of $V(G)\setminus V(P)$
has at most one neighbor in P.
and one can find an odd
hole of length ≥ 7.



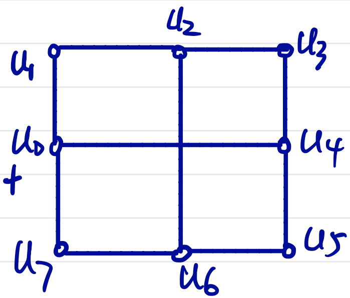
Then, we show that G does not induce P^-; the Petersen minus a vertex.

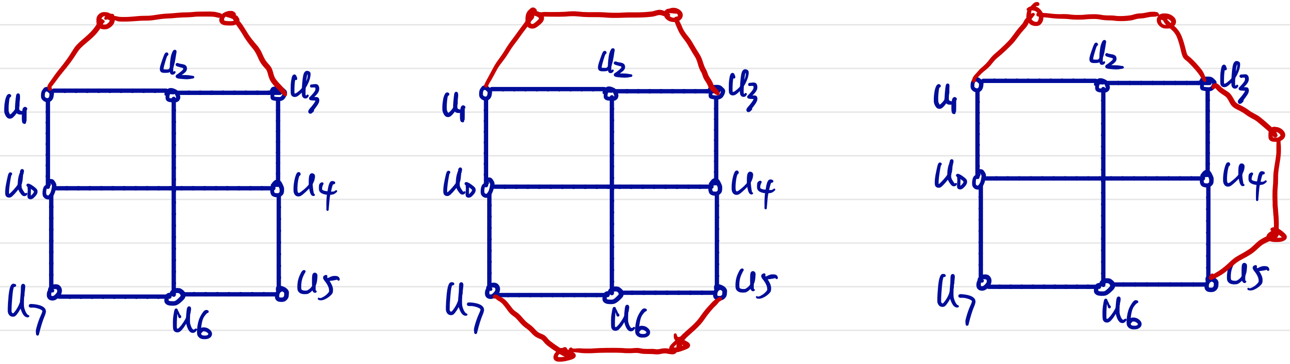
The argument is almost the same as above: every vertex of $V(G)\setminus V(P^-)$ has at most one neighbor in P^-; $G \neq P^-$ as G is 3-connected, and one can find an odd hole of length ≥ 7.



If G induces a Θ^+ then it must induce one of the following three configurations.

Θ^+





We can always find an odd hole of length ≥ 7.

Assuming an afirmative answer to the 3rd question of Plummer and Zha, i.e.,

every graph in $\mathcal{G}_2$ is 3-colorable,

perhaps it is true that

all graphs in $\bigcup_{\ell \geq 2} \mathcal{G}_\ell$ are 3-colorable.

Thank you

for your attendance!

Welcome to Nanjing