On sum of powers of graph eigenvalues: problems and progress

Huiqiu Lin
East China University of Science and Technology

May 27, 2021

Outline

1. Bounds for the sum of squares of positive eigenvalues of a graph
2. The Bollobás-Nikiforov's conjecture
3. The Brouwer's conjecture

Let G be a simple and undirected graph with n vertices, m edges, chromatic number χ and adjacency matrix A with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The inertia of A is the ordered triple (n^+, n_0, n^-), where n^+, n^- and n_0 are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of A respectively. Let $s^+ = \sum_{i=1}^{n^+} \lambda_i^2$ and $s^- = \sum_{i=n-n^-+1}^{n} \lambda_i^2$.

In 2016, Elphick, Farber, Goldberg and Wocjan posed the following conjecture.

Conjecture 1 (2016, DM)

Let G be a connected graph. Then

$$s^+ \leq 2m - n + 1.$$

The motivation of the above conjecture is from Hong's inequality.

Theorem (Hong, 1988, LAA)

Let G be a connected simple graph with m edges and n vertices. Then $\lambda_1^2(G) \leq 2m - n + 1$.

[Y. Hong, A bound on the spectral radius of graphs, Linear Algebra Appl. 108 (1988) 135–140.]
[C. Elphick, M. Farber, F. Goldberg, P. Wocjan, Conjectured bounds for the sum of squares of positive eigenvalues of a graph, Discrete Math. 339 (2016), no. 9, 2215–2223.]

Sketch of the proof of Hong's inequality.

Let $X = (x_1, \cdots, x_n)^t$ be the perron vector ($\sum_{i=1}^n x_i^2 = 1$) of $A(G)$ and let $X(i)$ denote the vector obtained from X by replacing x_j with 0 if v_i is not adjacent to v_j. Note that $AX = \lambda_1 X$, then

$$\lambda_1 x_i = (AX)_i = A_i X = A_i X(i)$$

where A_i denotes the ith row of A. Hence by the Cauchy-Schwartz inequality, we have

$$\lambda_1^2 x_i^2 = (A_i X(i))^2 \leq ||A_i||_2^2 ||X(i)||_2^2 = d_i(1 - \sum_{v_i \nsim v_j} x_j^2).$$

Sketch of the proof of Hong's inequality.

Taking a summation on i, we obtain that

$$\begin{aligned}
\lambda_1^2 &\leq 2m - \sum_{i=1}^{n} d_i \sum_{v_i \nsim v_j} x_j^2 \\
&= 2m - \sum_{i=1}^{n} d_i x_i^2 - \sum_{i=1}^{n} d_i \sum_{v_i \nsim v_j, i \neq j} x_j^2 \\
&\leq 2m - \sum_{i=1}^{n} d_i x_i^2 - \sum_{i=1}^{n} \sum_{v_i \nsim v_j, i \neq j} x_j^2 \\
&= 2m - \sum_{i=1}^{n} d_i x_i^2 - \sum_{i=1}^{n} (n - d_i - 1) x_i^2 \\
&= 2m - n + 1.
\end{aligned}$$

Sketch of the proof of Conjecture 1 for regular graphs.

- Ando and Lin showed that $\chi(G) \geq \max\{1+\frac{s^+}{s^-}, 1+\frac{s^-}{s^+}\}$.
- Brooks Theorem: If G is a connected graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.
- Conjecture 2 is equivalent to $s^- \geq n-1$ since $s^+ + s^- = 2m$.
- Let G be a k-regular connected graph and neither an odd cycle nor a complete graph. Then

$$s^- \geq \frac{s^+ + s^-}{\chi(G)} \geq \frac{2m}{k} = n.$$

[T. Ando, M. Lin, Proof of a conjectured lower bound on the chromatic number of a graph, Linear Algebra Appl. 485 (2015) 480–484.]

Let A be the adjacency matrix of a graph G of order n. We list the eigenvalues of A as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.
In 2007, Bollobás and Nikiforov posed the following conjecture.

Conjecture 2 (Bollobás, Nikiforov, JCTB, 2007)

Let G be a K_{r+1}-free graph of order at least $r+1$ with m edges. Then

$$\lambda_1^2 + \lambda_2^2 \leq \frac{r-1}{r} 2m.$$

[B. Bollobás, V. Nikiforov, Cliques and the spectral radius, *J. Combin. Theory Ser. B* **97** (2007), no. 5, 859–865.]

The Bollobás-Nikiforov conjecture is motivated by the following result which is obtained by Nikiforov.

Theorem (Nikiforov, CPC, 2002)

Let G be a K_{r+1}-free graph of order at least $r+1$ with m edges. Then

$$\lambda_1^2 \leq \frac{r-1}{r} 2m.$$

[V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, Combin. Probab. Comput. 11 (2002), no. 2, 179–189.]

Sketch of the proof.

- The Motzkin-Straus inequality: Let G be a graph on n vertices with $\omega(G) \leq r$. For any n-vector $(x_1, x_2, \ldots, x_n)$ with $x_i \geq 0$ $(1 \leq i \leq n)$ and $x_1 + x_2 + \cdots + x_n = 1$,

$$\sum_{i \sim j} x_i x_j \leq \frac{r-1}{r}.$$

- Let $y = (y_1, \ldots, y_n)$ be the perron vector of G (i,e, $y_i > 0$, $\sum_{i=1}^{n} y_i^2 = 1$).
- By the Cauchy inequality,

$$\lambda_1^2 = (\sum_{i \sim j} y_i y_j)^2 \leq 2m \sum_{i \sim j} y_i^2 y_j^2 \leq \frac{r-1}{r} 2m.$$

Def. 1

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$ for $i = 1, \ldots, n-1$. If

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n,$$

then we say that x is weakly majorized by y and denote $x \prec_w y$. If $x \prec_w y$, and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then we say that x is majorized by y and denote $x \prec y$.

For example, if $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = n$, then

$$(1, 1, \ldots, 1) \prec (x_1, x_2, \ldots, x_n) \prec (n, 0, \ldots, 0).$$

Def. 2

A square nonnegative matrix is called *doubly stochastic* if the sum of the entries in every row and every column is 1.

Def. 3

A nonnegative square matrix is called *doubly substochastic* if the sum of the entries in every row and every column is less than or equal to 1.

Def. 4

A square matrix is called a *weak-permutation matrix* if every row and every column has at most one nonzero entry and all the nonzero entries (if any) are 1.

Doubly stochastic matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

Doubly substochastic matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$

Weak-permutation matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

We first introduce three useful lemmas from Zhan.

Lemma 1

Every doubly stochastic matrix is a convex combination of permutation matrices.

Lemma 2

Every doubly substochastic matrix is a convex combination of weak-permutation matrices.

X. Zhan, Matrix theory, Graduate Studies in Mathematics, 147. American Mathematical Society, Providence, RI, 2013. x+253 pp.

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x = (x_1, \ldots, x_n), x_i \geq 0, 1 \leq i \leq n\}$.

Lemma 3

Let $x, y \in \mathbb{R}^n_+$. Then $y \prec_w x$ if and only if there exists a doubly substochastic matrix A such that $y = Ax$.

By the above lemmas, we get the following simple but very useful result.

Theorem 1 (L., Ning and Wu, CPC, 2021)

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be two n-nonnegative vectors with $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ and $y_1 \geq y_2 \geq \ldots \geq y_n \geq 0$. If $y \prec_w x$, then $\|x\|_p \geq \|y\|_p$ for $p > 1$ with equality holding if and only if $x = y$.

Sketch of the proof.

- Since $y \prec_w x$, there exist a doubly substochastic matrices A such that $y = Ax$ by Lemma 3.
- By Lemma 2, there exist weak-permutation matrices P_i for $i = 1, \cdots, n$, such that $A = \sum_{i=1}^{n} a_i P_i$ with $\sum_{i=1}^{n} a_i = 1$.
- $y = Ax = (\sum_{i=1}^{n} a_i P_i)x = \sum_{i=1}^{n} a_i (P_i x)$. Thus,

$$\begin{aligned} \|y\|_p &= \|\sum_{i=1}^{n} a_i (P_i x)\|_p \leq \sum_{i=1}^{n} a_i \|P_i x\|_p \\ &\leq \sum_{i=1}^{n} a_i \|x\|_p = \|x\|_p \sum_{i=1}^{n} a_i \\ &= \|x\|_p. \end{aligned}$$

Bollobás-Nikiforov conjecture is equivalent to if $\lambda_1^2 + \lambda_2^2 > \frac{r-1}{r} 2m$, then $K_{r+1} \subseteq G$. We first try $r = 2$ and get the following result.

Theorem 2 (L., Ning and Wu, CPC, 2021)

Let G be a graph of size m. If $\lambda_1^2 + \lambda_2^2 \geq m$, then G contains a triangle, unless G is a blow-up of some member of $\mathcal{G}$, where $\mathcal{G} = \{P_2 \cup K_1, 2P_2 \cup K_1, P_4 \cup K_1, P_5 \cup K_1\}$.

Sketch of the proof.

Let (n^+, n_0, n^-) be the inertia of $A(G)$.

- $t(G) = \frac{\lambda_1^3+\lambda_2^3+\cdots+\lambda_{n^+}^3+\lambda_{n-n^-+1}^3+\cdots+\lambda_n^3}{6}$.
- $\lambda_1^2 + \cdots + \lambda_{n^+}^2 + \lambda_{n-n^-+1}^2 + \cdots + \lambda_n^2 = 2m$.
  Note that $\lambda_1^2 + \lambda_2^2 \geq m \geq \lambda_{n-n^-+1}^2 + \cdots + \lambda_n^2$.
- Choose
  $x = (\lambda_1^2, \lambda_2^2, 0, \ldots, 0)^t, \quad y = (\lambda_n^2, \lambda_{n-1}^2, \ldots, \lambda_{n-n^-+1}^2)^t$, then
  $y \prec_w x$.
- By Theorem 1, we have $||x||_{\frac{3}{2}}^{\frac{3}{2}} \geq ||y||_{\frac{3}{2}}^{\frac{3}{2}}$.

Corollary (Noval 1970, Nikiforov 2009)

Let G be a graph of size m. If $\lambda_1^2 \geq m$, then G contains a triangle, unless G is a blow up of $P_2 \cup K_1$.

[E. Nosal, Eigenvalues of graphs, Calgary: Department of Mathematics of University of Calgary, 1970.]
[V. Nikiforov, More spectral bounds on the clique and independence numbers, *J. Combin. Theory Ser. B*, **99** (2009), no. 6, 819–826.]

- The Laplacian matrix $L(G) = \Delta(G) - A(G)$, where $\Delta(G) = Tr(d_1, \ldots, d_n)$.
- Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq 0$ be the eigenvalues of $L(G)$.
- Let $d = (d_1, \ldots, d_n)$ and $\mu = (\mu_1, \ldots, \mu_{n-1}, 0)$.
- The Schur-Horn Dominance Theorem implies $d \prec \mu$.
- For a non-negative integral sequence d, we define its conjugate degree sequence as the sequence $d' = (d'_1, \ldots, d'_n)$ where $d'_k := |\{i : d_i \geq k\}|$.

Gale-Ryser Theorem

There exists a $(0,1)$-matrix A with row and column sum vectors r and c if and only if $r \prec c'$.

Applying this to the adjacency matrix of G immediately gives that $d \prec d'$. In 1994, Grone and Merris raised the natural question whether the Laplacian spectrum sequence and the conjugate degree sequence are majorization comparable.

Grone-Merris Conjecture

For any graph G, the Laplacian spectrum is majorized by the conjugate degree sequence $\mu \prec d'$.

[R. Grone and R. Merris, Coalescence, majorization, edge valuations and the Laplacian spectra of graphs, Linear and Multilinear Algebra 27, No.2 (1990) 139–146.]
[R. Grone and R. Merris, The Laplacian spectrum of a graph II, SIAM J. Disc. Math. 7 (1994) 221–229.]

In 2011, Bai confirmed the Grone-Merris Conjecture.

Theorem (Bai, 2011)

For any graph G, the Laplacian spectrum is majorized by the conjugate degree sequence

$$\mu \prec d'.$$

Grone-Merris-Bai Theorem states that

$$S_k(G) = \sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} d_i'.$$

[H. Bai, The Grone-Merris conjecture, Trans. Amer. Math. Soc. 363 (2011), no. 8, 4463–4474.]

Brouwer proposed the following conjecture, which can be seen as a variation of Grone-Merris-Bai Theorem

Conjecture 3 (Brouwer, 2012)

For any graph G on n vertices and for each $k \in \{1, 2, \ldots, n\}$,

$$S_k(G) \leq e(G) + \binom{k+1}{2}.$$

[A.E. Brouwer, W.H. Haemers, Spectra of graphs, Springer, New York, 2012.]

Thank you for your attention!