Quasirandom combinatorial structures

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Motivation

- What does it mean to look randomly?
  graphs, permutations, etc.
  this talk: approach on substructure density
- computer science
  derandomization, cryptography
- statistics
  independance of data

Talk overview

- **Quasirandom graphs**
- Quasirandom tournaments
- **Quasirandom permutations**
- Quasirandom Latin squares

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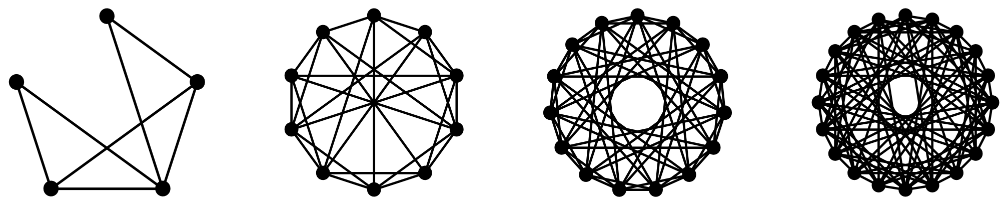
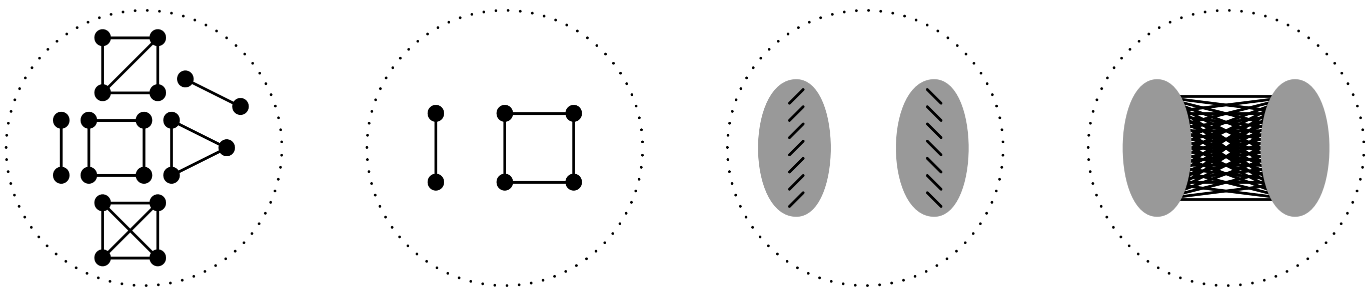
Classical results

- quasirandom graph ≈ Erdős-Rényi graph $G_{n,p}$
  not a property of a single graph but a sequence
- Rödl, Thomason, Chung, Graham and Wilson (1980's)
- $d(H,G)$ = (homomorphic) density of H in G
- $G_1, G_2, \ldots$ is quasirandom if $d(H,G_i) \to \mathbb{E}\, d(H,G_{n,p})$



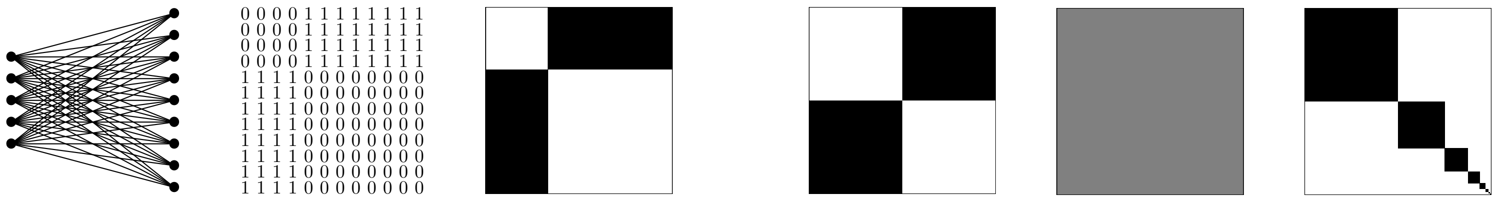
Equivalent characterizations

- $G_1, G_2, \ldots$ is quasirandom if $d(H, G_i) \to \mathbb{E}\, d(H, G_{n,p})$
  $\Leftrightarrow$ $d(K_2, G_i) \to p$ and $d(C_4, G_i) \to p^4$
  $\Leftrightarrow$ every n-vertex subset induces $\approx pn^2/2$ edges
  $\Leftrightarrow$ number of edges between A and B is $\approx p\,|A|\,|B|$
  $\Leftrightarrow$ spectrum of the adjacency matrix is $\{pn, o(n), \ldots,\}$



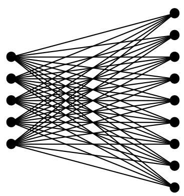
Graph limit view

- a sequence G_i is convergent if $d(H, G_i)$ converges
  quasirandom $\Leftrightarrow d(H, G_i) \to \mathbb{E}\, d(H, G_{n,p})$
- graphon analytic representation of the limit
  $W : [0,1]^2 \to [0,1]$, a "continuous" adjacency matrix
  regularity decompositions, martingale convergence
- possible to define $d(H, W)$ for every graph H

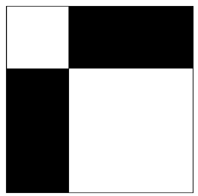


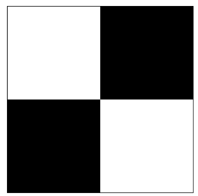
Graph limit view

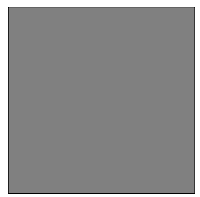
- a sequence G_i is convergent if $d(H, G_i)$ converges
- graphon analytic representation of the limit
  $W : [0,1]^2 \to [0,1]$, a "continuous" adjacency matrix
  density $d(H, W)$ of a graph H in W
- a sequence G_i is quasirandom iff $W = 1/2$ a.e.
  $d(K_2, W) = p$ and $d(C_4, W) = p^4 \Leftrightarrow W = p$

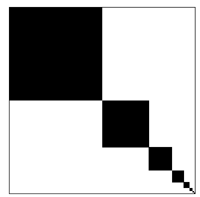


$$\begin{matrix} 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0 \end{matrix}$$



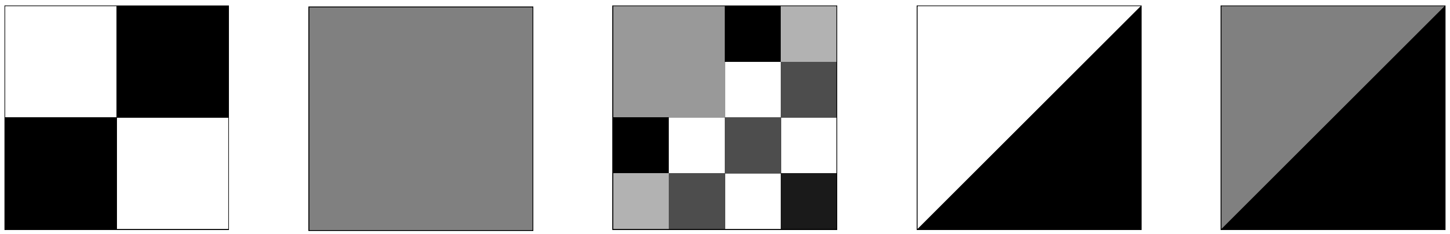






Finitely forcible graph limits

- $d(K_2, W) = p$ and $d(C_4, W) = p^4 \Leftrightarrow W = p$
- a graphon W_0 is finitely forcible
  if $\exists G_i, d_i$ s.t. $d(G_i, W) = d_i \Leftrightarrow W = W_0$
- another example: $d(K_2, W) = 1/2$ and $d(K_3, W) = 0$
- Simple structure? Useful for extremal graph theory?

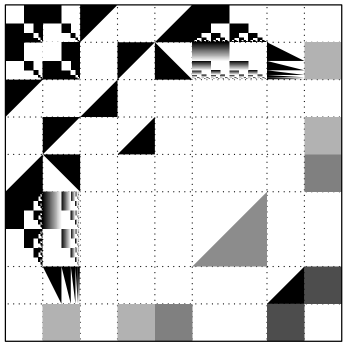
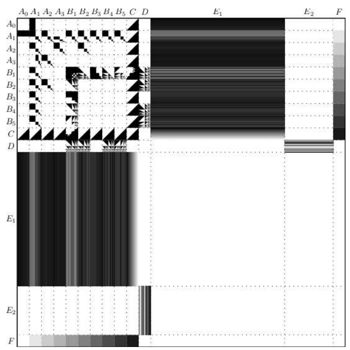
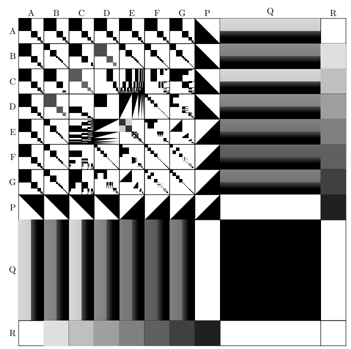
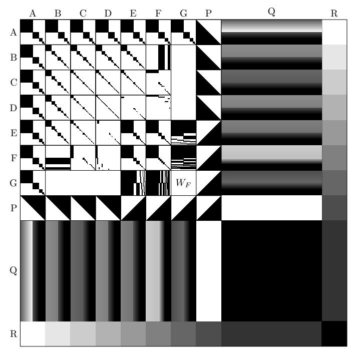


Simple structure?

- Conjectures (Lovász and Szegedy, 2011):
  The space $T(W)$ of a finitely forcible W is compact.
  The space $T(W)$ has finite dimension.
- Theorem (Glebov, K., Volec, 2019):
  $T(W)$ can fail to be locally compact
- Theorem (Glebov, Klimošová, K., 2019):
  $T(W)$ can have a part homeomorphic to $[0,1]^\infty$
- Theorem (Cooper, Kaiser, K., Noel, 2018):
  $\exists$ finitely forcible W with no small ε-regular partition

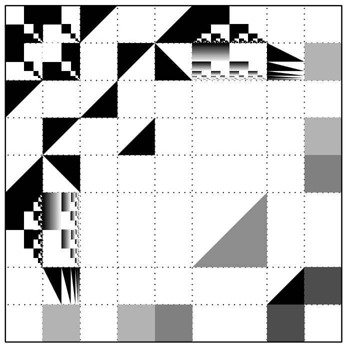
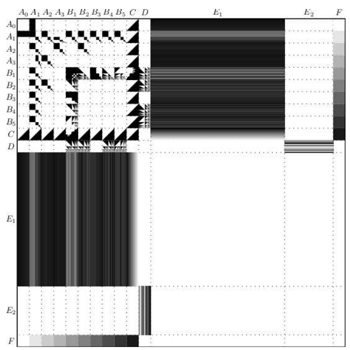
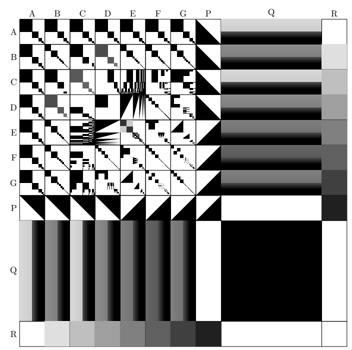
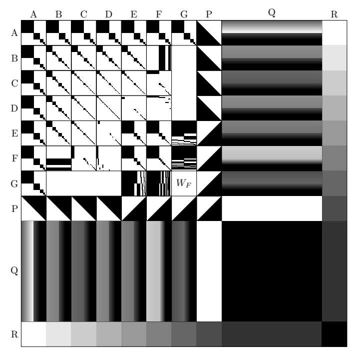
Universality

- Theorem (Cooper, K. Martins, 2018)
  Every graphon can be embedded
  in a finitely forcible graphon.
- Theorem (K., Lovász Jr., Noel, Sosnovec, 2020)
  The embedding can be made to be of $1 - \varepsilon$ of the host.

Extremality

- Every finitely forcible graphon is extremal.
- Conjecture (Lovász and Szegedy, 2011):
  Every problem has a finitely forcible optimal solution.
- Theorem (Grzesik, K., Lovász Jr., 2020)
  Extremal problems with no finitely forcible optimum.

Talk overview

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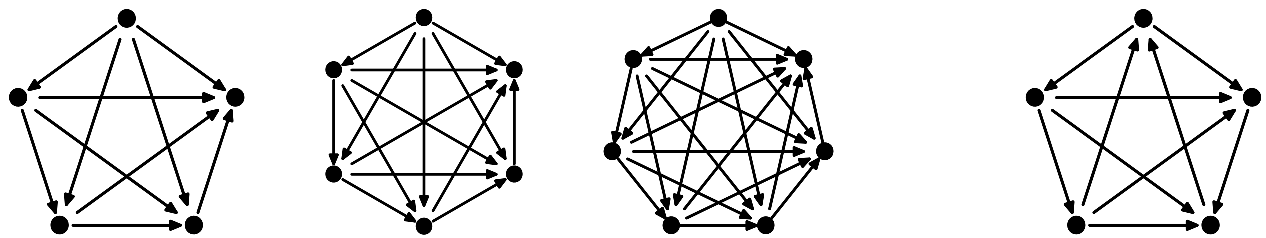
- Quasirandom tournaments
- Quasirandom permutations
- Quasirandom Latin squares

Which tournaments force?

- tournament is an orientation of a complete graph
- every transitive tournament is quasirandom-forcing
- additional 5-vertex (Coregliano, Parente, Sato, 2019)
- no $\geq$ 7-vertex (Bucić, Long, Shapira, Sudakov, 2019+)
- no additional at all tournaments (Hancock, Kabela, K., Martins, Parente, Skerman, Volec, 2019+)



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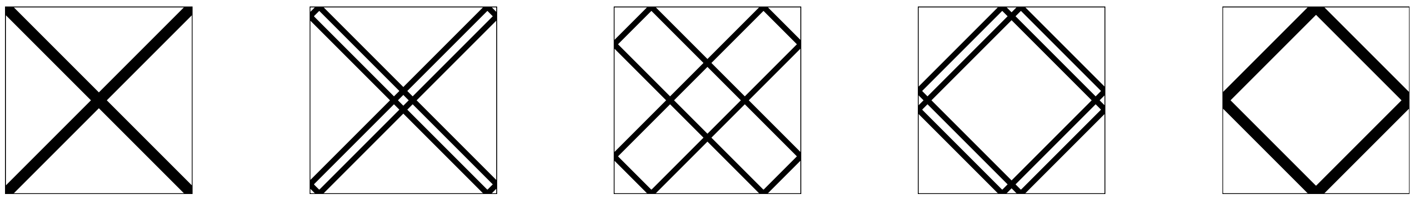
Permutation limits

- permutation of order n: order on numbers $1, \dots, n$
  subpermutation: $4\underline{53}21\underline{6} \rightarrow 213$ $\quad$ $4\underline{53}2\underline{1}6 \rightarrow 321$
- probability measure μ on $[0, 1]^2$ with unit marginals
  Hoppen, Kohayakawa, Moreira, Ráth and Sampaio
  similar ideas in work of Presutti and Stromquist
- μ-random permutation
  choose n random points, x- and y-coordinates

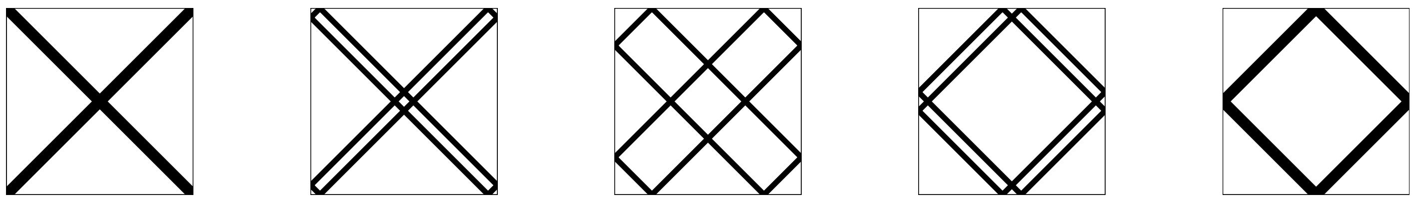
QUASIRANDOM PERMUTATIONS

- a sequence Π_i is quasirandom
  $\Leftrightarrow d(\pi, \Pi_i) \to 1/k!$ for every $\pi \in S_k$ and all k
  $\Leftrightarrow \Pi_i$ converges to the uniform measure
- Question (Graham)
  Does there exist k_0 such that quasirandomness
  $\Leftrightarrow d(\pi, \Pi_i) \to 1/k_0!$ for every $\pi \in S_{k_0}$?
- $k_0 = 3$ is not sufficient: $d(123, .)$ ranges from $1/4$ to $1/8$



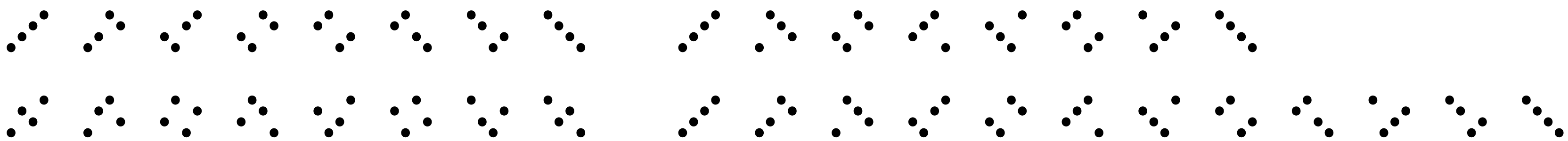
Quasirandom forcing

- Question (Graham)
  quasirandomness $\Leftrightarrow d(\pi,\Pi_i)\to 1/k_0!$ for every $\pi\in S_{k_0}$
- Theorem (K., Pikhurko, 2013)
  quasirandomness $\Leftrightarrow d(\pi,\Pi_i)\to 1/24$ for every $\pi\in S_4$
  independence tests (Hoeffding 1948, Yanagimoto 1970)
- Do we need that $d(\pi,\Pi_i)=1/24$ for all $\pi\in S_4$?



Quasirandom forcing

- Do we need that $d(\pi, \Pi_i) = 1/24$ for all $\pi \in S_4$?
- Theorem (Kurečka, 2020+)
  At least 4 permutations (regardless of orders) needed.
- Theorem (Chan, K., Noel, Pehova, Sharifzadeh, Volec)
  characterization of sets $T \subseteq S_4$ such that
  Π_i is quasirandom $\Leftrightarrow \sum_{\pi \in T} d(\pi, \Pi_i) \to |T|/24$



or complement of one of these four sets

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Latin squares

- Latin square
  each row/column contain all numbers $1, \ldots, n$
- pattern density: choose rows and columns
- limit theory by Garbe, Hancock, Hladký, Sharifzadeh
  sampling is tricky (existence of designs)

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 5 & 1 & 4 \\ 5 & 4 & 2 & 3 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$$

Quasirandom Latin squares

- Conjecture (Garbe, Hancock, Hladký, Sharifzadeh)
  quasirandomness $\Leftrightarrow$ density of $k \times \ell$ pattern is $1/(k\ell)!$
- Theorem (Cooper, K., Lamaison, Mohr, 2020+)
  quasirandomness $\Leftrightarrow$ density of 2×3 pattern is $1/720$
  2×3 cannot be replaced with $1 \times \ell$ or 2×2

1	2	3	4	5
3	1	4	5	2
4	5	1	2	3
2	3	5	1	4
5	4	2	3	1

$\rightarrow$

1	3
2	4

Open problems

- minimal quasirandom forcing subsets of S_4
- minimal quasirandom forcing sets of permutations
  Can four permutation force quasirandomness?
- general theory of quasirandom relational structures??
  A common property of finiteness characterizations

Thank you for your attention!