Rainbow matchings for 3-uniform hypergraphs

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June 3, 2021
SCMS Seminar

Erdős Matching Conjecture

Erdős Matching Conjecture, 1965

For positive integers k, n, s, if H is a k-graph on n vertices and $\nu(H) < s$, then

$$e(H) \leq \max\left\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\right\}.$$

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Theorem (Erdős, 1965)

The conjecture is true for $n > n_0(k, s)$.

Erdős Matching Conjecture

- Frankl, 2013
  The conjecture is true for $n > (2s-1)k - s$.
- Frankl and Kupavskii, 2018
  The conjecture is true for $s \geq s_0$ and $n \geq \frac{5}{3}(s-1)k - \frac{2}{3}(s-1)$.

For 3-graphs:

- Frankl, 2017
  The conjecture is true for $k = 3$.

Rainbow Matching

Conjecture (Huang, Loh and Sudakov, 2012; independently by Aharoni and Howard)

Let $F_1, \ldots, F_t$ be k-graphs on $[n]$. If

$$|F_i| > \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$$

for all $1 \leq i \leq t$, then there is a 'rainbow' matching of size t: one that contains exactly one edge from each family.

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Theorem (Huang, Loh and Sudakov, 2012)

The conjecture above is true for $n > 3k^2t$.

Rainbow Matching

Conjecture

Let $F_1, \ldots, F_t$ be k-graphs on $[n]$. If

$$\delta_1(F_i) > \binom{n-1}{k-1} - \binom{n-t}{k-1}$$

for all $1 \leq i \leq t$, then there is a 'rainbow' matching of size t.

Definition

$\delta_1(H)$: minimum vertex degree in H

Dirac Type Problem

Theorem (Dirac, 1952)

A simple graph G with n vertices ($n \geq 3$) is Hamiltonian if $\delta(G) \geq n/2$. In particular, if n is even, then G contains a perfect matching.

Theorem (Kühn, Osthus and Treglown, 2013; independently, Khan, 2013)

There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-uniform hypergraph whose order $n > n_0$ is divisible by 3. If

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$$

then H has a perfect matching.

Our Result

Theorem (Lu, Yu, and Y., 2021)

Let $n \in 3\mathbb{Z}$ be sufficiently large, and let $\mathcal{F} = \{F_1, \ldots, F_{n/3}\}$ be a family of 3-graphs such that $|V(F_i)| = n$ and $V(F_i) = V(F_1)$ for $i \in [n/3]$. If

$$\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$$

for $i \in [n/3]$, then $\mathcal{F}$ admits a rainbow matching.

Proof Idea

- We convert it to the problem of finding a perfect matching in a balanced $(1, 3)$-partite 4-graph.
- For any integer $k \geq 3$, a k-graph H is $(1, k-1)$-partite if there exists a partition of $V(H)$ into sets V_1, V_2 (called *partition classes*) such that for any $e \in E(H)$, $|e \cap V_1| = 1$ and $|e \cap V_2| = k - 1$.
- A $(1, k-1)$-partite k-graph with partition classes V_1, V_2 is balanced if $(k-1)|V_1| = |V_2|$.

Proof Idea

- We define a $(1,3)$-partite 4-graph H with respect to $\mathcal{F}$ by letting
  - $V(H) = P \cup Q$, where $P = V(F_i)$ and $Q = \{v_1, \ldots, v_{n/3}\}$, and
  - $N_H(v_i) = F_i$, for each $i \in [n/3]$.
- Then we have

$$\delta_1(N_H(v_i)) > \binom{n-1}{2} - \binom{2n/3}{2}$$

for each $i \in [n/3]$.

Proof Idea

For the case when H is close to H_0, we prove the conjecture is true for $k = 3$ and all $t \in [n/3]$ (by induction).

- Base: $t \leq n/200$, greedy construction
- $t \geq n/400$
  - ◇ H is close to H_0 at every vertex
  - ◇ H is not close to H_0 at some vertices

Proof Idea

When H is not close to H_0, we follow an approach by Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov.

- Find an absorber M_{abs}
- Find an almost perfect matching M' in $H \setminus V(M_{abs})$
  - ◇ Find perfect fractional matchings in random subgraphs of $H - V(M_{abs})$
  - ◇ Convert to an almost perfect matching
- Use M_{abs} to extend M' to a perfect matching

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Absorbing Lemma

Lemma

Let $n \in 3\mathbb{Z}$ be large enough and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P|$ and

$$\delta_1(H) \geq \frac{n}{3}\left(\binom{n-1}{2} - \binom{2n/3}{2} + 1\right).$$

Let ρ, ρ' be constants such that $0 < \rho' \ll \rho \ll 1$. Then H has a matching M' such that $|M'| \leq \rho n$ and, for any subset $S \subseteq V(H)$ with $|S| \leq \rho' n$ and $3|S \cap Q| = |S \cap R|$, $H[S \cup V(M)]$ has a perfect matching.

Proof Idea

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Almost Perfect Matching

- Form a random subgraph $R \subseteq H$ by taking each vertex with probability $n^{-0.9}$. We take $n^{1.1}$ independent copies of R.
- With high probability, all those copies have certain properties, for example, containing a perfect fractional matching.
- Those properties enable us to perform another round of random sampling to find a spanning subgraph satisfying the conditions of a 'Rödl Nibble' theorem .

Proof Idea

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- Find an absorber M_{abs}
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Fractional Matching

Lemma

Let ρ, ε be constants with $0 < \varepsilon \ll 1$ and $0 < \rho < \varepsilon^{12}$, and let H be a $(1,3)$-partite 4-graph with partition classes Q, P (with $3|Q| = |P|$). Suppose

$$d_H(\{u,v\}) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$$

for any $v \in Q$ and $u \in P$. If H contains no independent set S with $|S \cap Q| \geq n/3 - \varepsilon^2 n$ and $|S \cap P| \geq 2n/3 - \varepsilon^2 n$, then H contains a perfect fractional matching.

Fractional Matching

- We bound the size of independent sets in each random induced subgraph using a hypergraph container result. ("not close to the extremal graph")
- Using the Strong Duality Theorem, we would like to convert this problem to finding a perfect matching in a stable family, and deal with it by Tutte-Berge formula.

Fractional Matching

- We bound the size of independent sets in each random induced subgraph using a hypergraph container result. (“not close to the extremal graph”)
- Using the Strong Duality Theorem, we would like to convert this problem to finding a perfect matching in a stable family, and deal with it by Tutte-Berge formula.
- So we need to take the first round of random sampling more carefully such that each induced subgraph taken is balanced.

Balanced Induced Subgraphs

Lemma

Let $S \subset V(H)$ be a set of vertices such that

$$|S \cap Q| = n^{0.99}/3 \quad \text{and} \quad |S \cap P| = n^{0.99}.$$

Let R_+^i be chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$, for each $i \in [n^{1.1}]$, independently.

Define $R_-^i = R_+^i \cap (V(H) \setminus S)$, $1 \leq i \leq n^{1.1}$.

Then, with probability $1 - o(1)$, there exist R_i, $i \in [n^{1.1}]$, such that $R_-^i \subseteq R^i \subseteq R_+^i$ and R^i is balanced.

Proof Idea

When H is not close to H_0, we follow an approach by Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov.

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Almost Perfect Matching

Theorem (Pippenger and Spencer, 1989)

For every integer $k \geq 2$ and real $r \geq 1$ and $a > 0$, there are $\gamma = \gamma(k, r, a) > 0$ and $d_0 = d_0(k, r, a)$ such that for every n and $D \geq d_0$ the following holds: Every k-uniform hypergraph $H = (V, E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:

(1) *For all vertices $x \in V$ but at most γn of them, $d(x) = (1 \pm \gamma)D$;*

(2) *For all $x \in V$, $d(x) < rD$;*

(3) *For any two distinct $x, y \in V$, $d(x, y) < \gamma D$;*

contains a cover of at most $(1 + a)(n/k)$ edges.

Almost Perfect Matching

Lemma

Let $\sigma > 0$ and $0 < \rho \leq \varepsilon/3 \ll 1$. Let H be a $(1,3)$-partite 4-graph with partition classes Q, P, where $|Q| = n/3$ and $|P| = n$. Suppose H is not ε-close to $H_{1,3}(n, n/3)$ and

$$d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$$

for any $v \in Q$ and $u \in P$. Then H contains a matching covering all but at most σn vertices.

Proof Idea

When H is not close to H_0, we follow an approach by Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov.

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Proof Idea

Thus the balanced $(1,3)$-partite 4-graph H defined by $\mathcal{F}$ has a perfect matching, and hence the following holds.

Theorem (Lu, Yu, and Y., 2021)

Let $n \in 3\mathbb{Z}$ be sufficiently large, and let $\mathcal{F} = \{F_1, \ldots, F_{n/3}\}$ be a family of 3-graphs such that $|V(F_i)| = n$ and $V(F_i) = V(F_1)$ for $i \in [n/3]$. If

$$\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$$

for $i \in [n/3]$, then $\mathcal{F}$ admits a rainbow matching.

Thank you!

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