Ramsey upper density of infinite graphs

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Introduction

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In a red-blue-coloring of K_n, we can always find a monochromatic path with more than $2n/3$ vertices (Gerencsér-Gyárfás, 1967), but cannot guarantee a clique larger than $\Theta(\log n)$ (Erdős, 1947).

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Let H be a countably infinite graph. We define its Ramsey upper density $\rho(H)$ as the supremum of the values of λ with the following property: in every red-blue coloring of $E(K_{\mathbb{N}})$, there exists a monochromatic copy H' of H with $\bar{d}(V(H')) \geq \lambda$.

$\rho(P_\infty)$

The Ramsey upper density, for the particular case $H = P_\infty$, was introduced by Erdős and Galvin.

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Theorem (Corsten-DeBiasio-L-Lang, 2019)

$$\rho(P_\infty) = \frac{12+\sqrt{8}}{17} = 0.87226\ldots$$

$\rho(H)$ for general H

Let H be a (locally finite) graph. What properties about H have the biggest influence in the value of $\rho(H)$?

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- $\rho(\text{infinite } k\text{-ary tree}) \overset{conj}{=} \frac{k+1}{2k}$ for $k \geq 3$.

Expansion of independent sets

Given $S \subseteq V(H)$, we define $N(S) = \bigcup_{v \in S} N(v)$. We define

$$\mu(n, H) = \min\{|N(I)| : I \subset V(H) \text{ independent}, |I| = n\}.$$

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$\frac{\mu(n,H)}{n}$ and $\rho(H)$ are related through a certain function, which satisfies

$$\frac{x+1}{2x+1} \leq f(x) \leq \begin{cases} \frac{2x^2+3x+7+2\sqrt{x+1}}{4x^2+4x+9} & \text{for} \quad 0 \leq x < 3, \\ \frac{x+1}{2x} & \text{for} \quad x \geq 3. \end{cases}$$

The upper bound is tight in $[0, 1]$, and conjectured to hold elsewhere.

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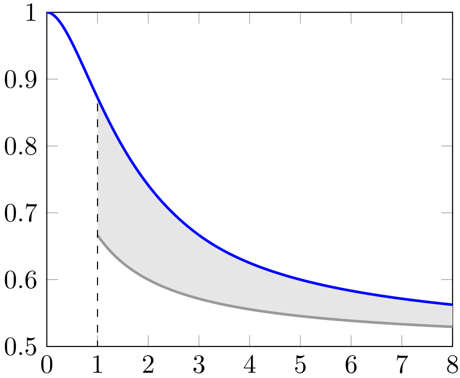

$f(x)$

Definition

Let $\gamma \in (-1, 1)$. For a continuous function $g(x) : [0, +\infty) \to \mathbb{R}$, define

$$\Gamma_\gamma^+(g, t) = \min\{x : \gamma x + g(x) \geq t\}$$

$$\Gamma_\gamma^-(g, t) = \min\{x : \gamma x - g(x) \geq t\},$$

where we take the minimum of the empty set to be $+\infty$. Let $h(\gamma)$ be the infimum, over all 1-Lipschitz functions g with $g(0) = 0$, of

$$h(\gamma) = \inf_g \limsup_{t\to\infty} \frac{\Gamma_\gamma^+(g, t) + \Gamma_\gamma^-(g, t)}{t}.$$

Define $f : [0, +\infty) \to \mathbb{R}$ as

$$f(\lambda) = 1 - \frac{1}{\frac{2\lambda}{(1+\lambda)^2} h\left(\frac{\lambda-1}{\lambda+1}\right) + \frac{2\lambda}{1+\lambda}}.$$

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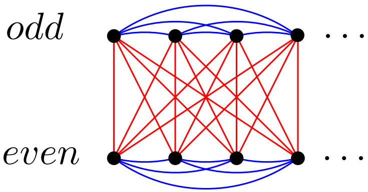

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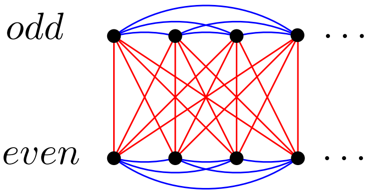

Theorem

Let H be a graph with $\chi(H) \geq a$, such that there is a finite set $S \subseteq V(H)$ with $V(H) \setminus S$ being contained in b components. Then $\rho(H) \leq b/(a-1)$.

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- *If H has finitely many components and finite chromatic number, let $a = \chi(H)$ and b be the number of infinite components of H. Then*

$$\min\{b/(2a-2), 1/2\} \leq \rho(H) \leq \min\{b/(a-1), 1\}.$$

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Conjecture (DeBiasio-McKenney, 2019)

For every $k \in \mathbb{N}$ there exists $c_k > 0$ such that $\rho(H) \geq c_k$ for every graph H with $\Delta(H) \leq k$.

Non-bipartite graphs

Theorem

Let H be a locally finite graph, a, b, r, s be positive integers with $a > b$, and $\Psi : V(H) \to [a]$ be a proper coloring. Suppose that there exist infinitely many pairwise disjoint independent sets $I_1, I_2, \dots$ in H, not concentrated in fewer than b components, such that $|I_i| = r$, $|N(I_i)| \leq s$, and $\Psi(N(I_i)) = 1$. Then

$$\rho(H) \geq \frac{b}{a-1} f\left(\frac{s}{r}\right).$$

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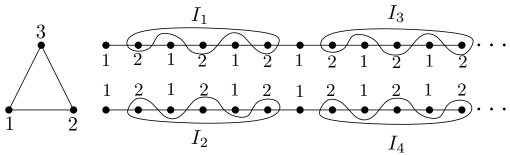

Results for infinite factors

Theorem

Let F be a finite graph and let I be an independent set such that $N(I)$ is independent. Then $\rho(\omega \cdot F) \geq f\left(\frac{|N(I)|}{|I|}\right)$.

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$$\rho(\omega \cdot F) \geq \frac{|V(F)|}{2|V(F)| - \alpha(F)} = \bar{f}\left(\frac{|V(F)|}{\alpha(F)} - 1\right), \quad \bar{f}(x) = \frac{x+1}{2x+1}.$$

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Corollary

For all $n \geq 2$, $\rho(\omega \cdot K_n) = f(n-1)$ *and* $\rho(\omega \cdot C_{2n-1}) = f\left(\frac{n}{n-1}\right)$.

Results for infinite factors

We have shown that $\rho(\omega \cdot K_3) = f(2)$, but we do not know the exact value of $f(2)$. From the bounds we know so far,

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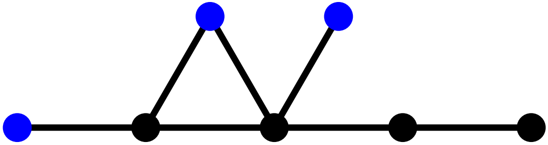

Sketch of the proof: upper bound

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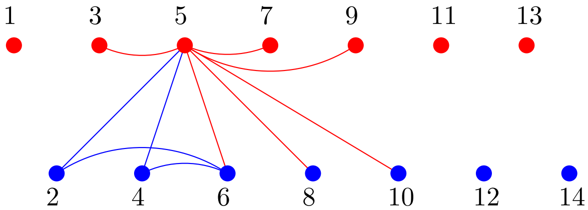

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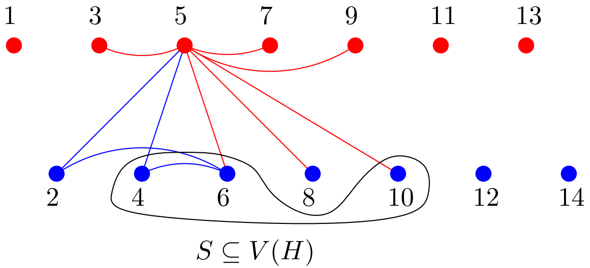

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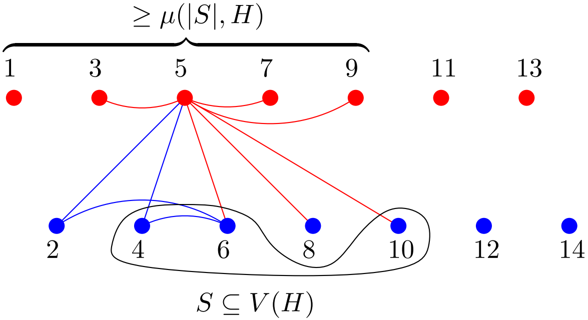

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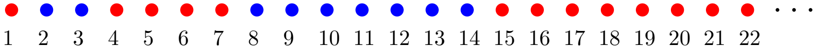

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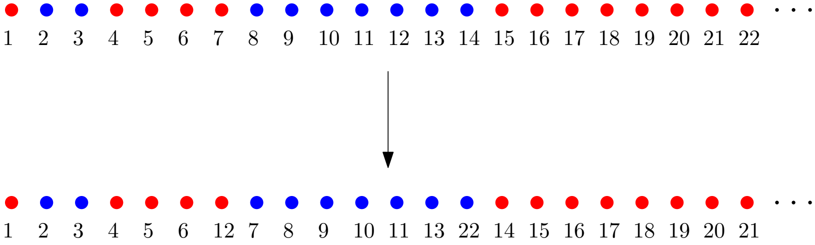

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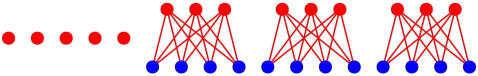
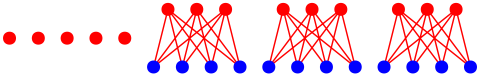
- There is a red-blue coloring of $V(K_{\mathbb{N}})$ such that, for every finite set S of vertices of color C there are infinitely many vertices joined to every element of S through edges of color C.
- There exists a subgraph $F \subseteq K_N$ with $\bar{d}(V(F)) \geq f(s/r)$, and a color C of the form below.



Sketch of the proof: lower bound

We will look at the bipartite case. Suppose that the coloring of $E(K_{\mathbb{N}})$ is given. We want to find a dense copy of H. We proceed in three steps:

- There is a red-blue coloring of $V(K_{\mathbb{N}})$ such that, for every finite set S of vertices of color C there are infinitely many vertices joined to every element of S through edges of color C.
- There exists a subgraph $F \subseteq K_N$ with $\bar{d}(V(F)) \geq f(s/r)$, and a color C of the form below.
- There is a copy of H in color C, with $\bar{d}(V(H')) \geq \bar{d}(V(F))$.



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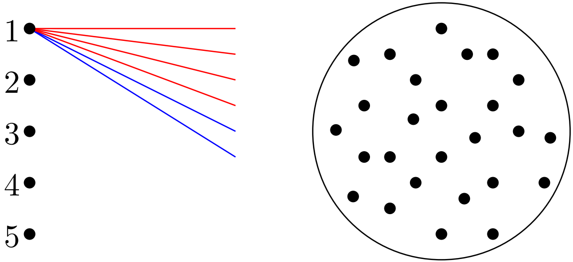

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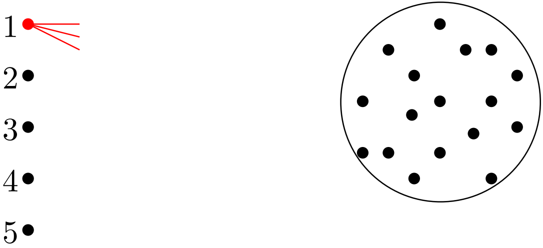

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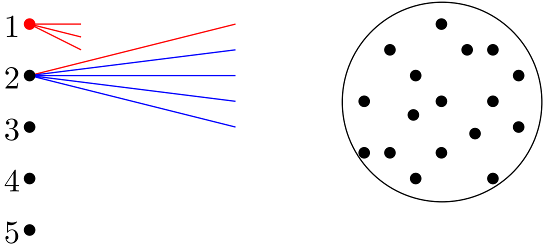

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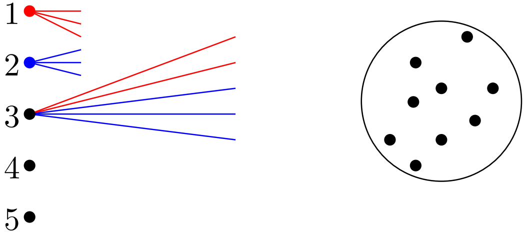

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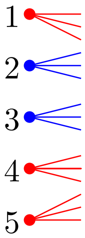

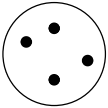

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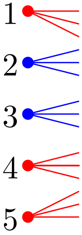
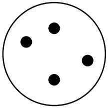
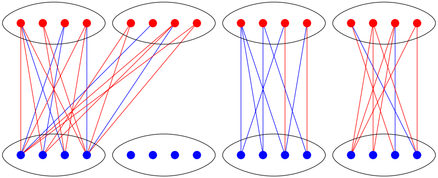




In this coloring, it is easy to put red vertices into a red copy of H, and vice versa.

Sketch of the proof: lower bound

Step 2: Apply Szemerédi's Regularity Lemma to finite but large pieces of $\mathbb{N}$. Use it to find copies of $K_{r,s}$.



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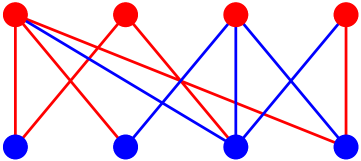

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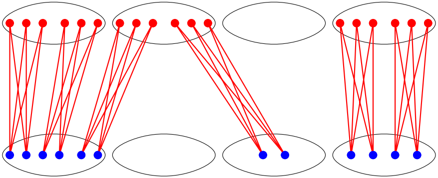

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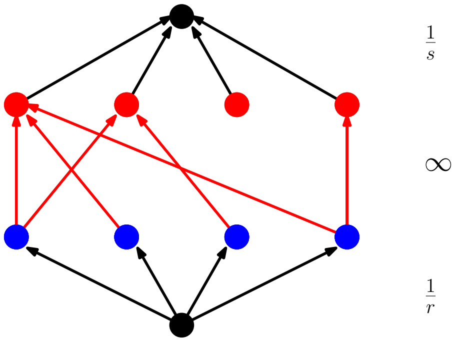

This reduces the $K_{r,s}$ packing problem to a max-flow problem.

Sketch of the proof: lower bound

Step 2: solve the max-flow $\rightarrow$ min-cut problem. By looking at the degree sequence in each color, we can further reduce the problem to the following lemma:

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Lemma

Let $g : [0, +\infty) \to [0, \infty)$ *be a non-decreasing continuous function. Let* $\lambda > 0$*. Define*

$$\ell_\lambda^+(g, t) = \min \{x : g(\lambda x) - x \geq t\},$$

$$\ell_\lambda^-(g, t) = \min \left\{x : x - \frac{g(x)}{\lambda} \geq t\right\},$$

where $\min \emptyset = +\infty$*. Then*

$$\limsup_{t \to \infty} \frac{\ell^+(g, t) + \ell^-(g, t)}{t} \geq \frac{f(\lambda)}{1 - f(\lambda)}$$

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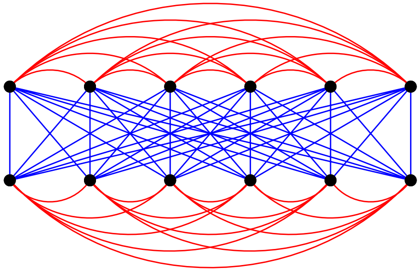
- Isolated vertices of F are mapped to individual vertices of H.
- $K_{r,s}$ components of F are mapped to sets of the form $I \cup N(I)$.
- Individual vertices v of H are mapped to either arbitrary vertices of color C, or common neighbors of the preimage of $N(v)$ through edges of color C.

What goes wrong when H is not bipartite?



Sketch of the proof: non-bipartite H

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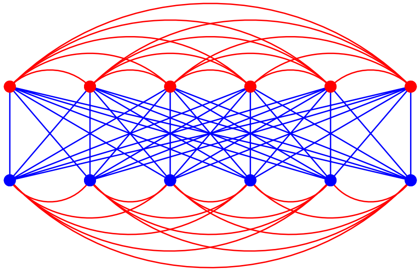

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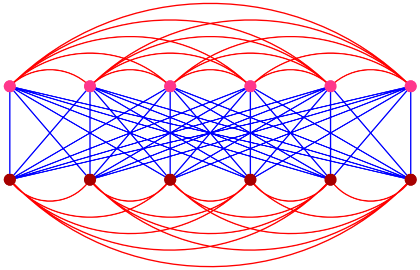

We can use a coloring of Elekes-Soukup-Soukup-Szentmiklóssy instead.

Linear search problem

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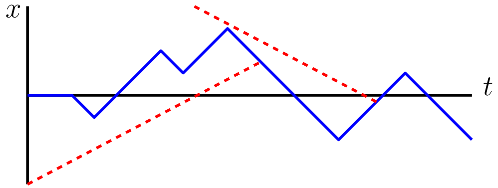

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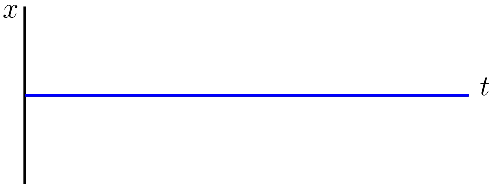

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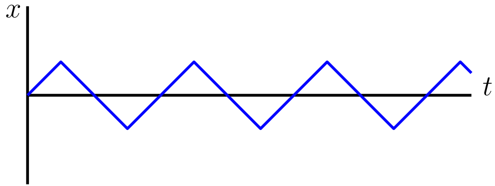

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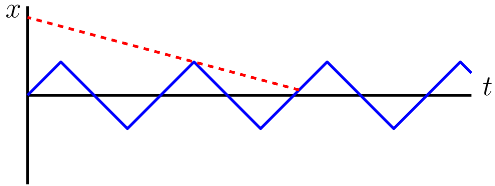

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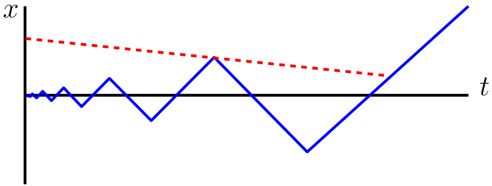

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Problem

For a fixed σ, minimize

$$\limsup_{s \to \infty} \frac{\max\{\Gamma_\sigma^+(g, s), \Gamma_\sigma^-(g, s)\}}{s}$$

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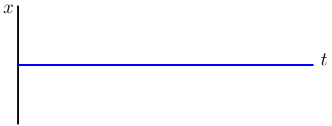

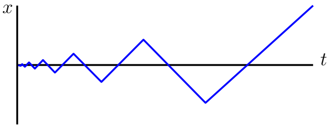

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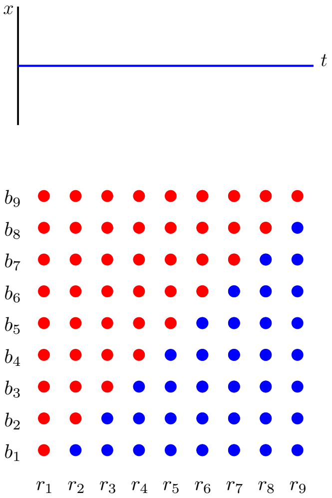

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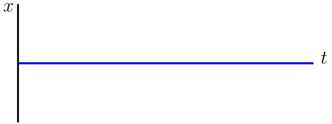

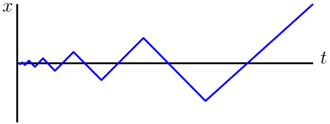

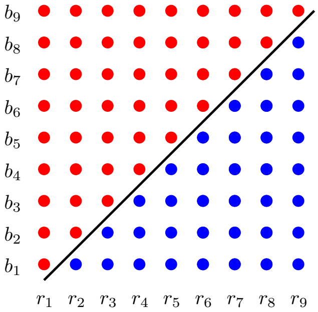

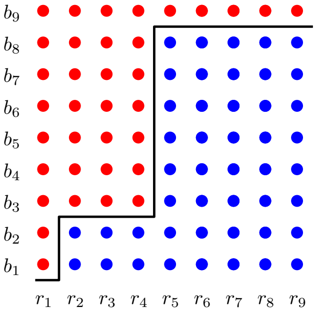

Open problem

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Problem

What is the value of $\rho(P_\infty^2)$? Improve either the previous bound or $\rho(P_\infty^2) \leq 1/2$.