# The homotopy type of the independence complex of ternary graphs

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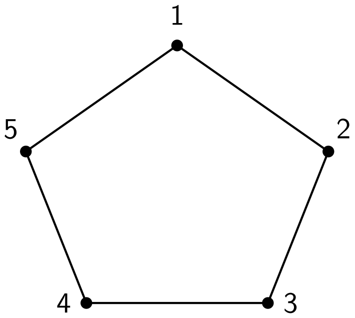
# Independence complex

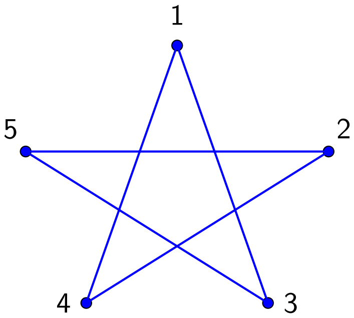
- A **graph** $G$ is an ordered pair of a vertex set $V(G)$ and an edge set $E(G)$, where an edge $e \in E(G)$ is an un-ordered pair of vertices.
- $I \subseteq V(G)$ is an **independent set** of a graph $G$ if any two vertices in $I$ does not form an edge of $G$.
- A **simplicial complex** on the ground set $V$ is a collection of subsets of $V$ such that it is closed under taking subsets.
- The **independence complex** of a graph $G$ is

$$I(G) = \{I \subseteq V(G) : I \text{ is an independent set of } G\}.$$

# Independence complexes of cycles




5-cycle $C_5$




$I(C_5) \simeq \mathbb{S}^1$

Kozlov(1999):

$$I(C_\ell) \simeq \begin{cases} \mathbb{S}^k \vee \mathbb{S}^k & \text{if } \ell = 3k+3, \\ \mathbb{S}^k & \text{if } \ell = 3k+2 \text{ or } 3k+4. \end{cases}$$

# Known results

**Theorem** (Ehrenborg-Hetyei, 2006)

If a graph $G$ does not contain a cycle, then $I(G)$ is either contractible or homotopy equivalent to a sphere.

A graph $G$ is **chordal** if it contains induced cycles of length at least $4$.

**Theorem** (Engström, 2008)

For a chordal graph $G$, $I(G)$ is either contractible of homotopy equivalent to the repeated wedge sum of a finite number of spheres.

# Kalai-Meshulam conjecture

A graph $G$ is **ternary** if it has no induced cycles of length divisible by $3$.

**Conjecture 1** (Kalai-Meshulam)

For a ternary graph $G$, the number of independent sets of even size and the number of independent sets of odd size differ by at most $1$.

For a simplicial complex $X$, let $f_i(X)$ be the number of $i$-dimensional faces of $X$.

**Conjecture 1 (reformulated)**

$|\sum_{i\geq 0}(-1)^i f_i(I(G))| \leq 1$ for every ternary graph $G$.

# Kalai-Meshulam conjecture

For a simplicial complex $X$, consider the chain complex

$$\cdots \xrightarrow{\partial_{i+2}} C_{i+1}(X) \xrightarrow{\partial_{i+1}} C_i(X) \xrightarrow{\partial_i} C_{i-1}(X) \xrightarrow{\partial_{i-1}} \cdots$$

where $C_i(X)$ is the $\mathbb{Z}$-module generated by the $i$-dimensional faces of $X$ for $i \geq 0$ and $C_{-1}(X) = \mathbb{Z}$.

- The **$i$-th (reduced) homology group** of $X$ is $\tilde{H}_i(X) = \text{ker}\partial_i/\text{im}\partial_{i+1}$.
- The **$i$-th (reduced) Betti number** of $X$ is $\tilde{\beta}_i(X) = \text{rank}\tilde{H}_i(X)$.

**Conjecture 2** (Kalai-Meshulam)

$\sum_{i\geq 0} \tilde{\beta}_i(I(G)) \leq 1$ for a ternary graph $G$.

**Observation**: $\sum_{i\geq 0}(-1)^i f_i(X) = \sum_{i\geq 0}(-1)^i \tilde{\beta}_i(X)$.

**Conjecture 2** $\implies$ **Conjecture 1**.

# Kalai-Meshulam conjecture

**Conjecture 3** (Engström)

For a ternary graph $G$, $I(G)$ is either contractible or homotopy equivalent to a sphere.

- $\tilde{\beta}_i(X) = 0$ for all $i \geq 0$ if $X$ is contractible.
- For a $d$-dimensional sphere $\mathbb{S}^d$, $\tilde{\beta}_d(\mathbb{S}^d) = 1$ and $\tilde{\beta}_i(\mathbb{S}^d) = 0$ if $i \neq d$.

**Conjecture 3** $\implies$ **Conjecture 2**.

# Main results

- Chudnovsky-Scott-Seymour-Spirkl (2020): proved **Conjecture 1**.
- Zhang-Wu (2020): proved **Conjecture 2**.
- Engström (2020): proved a weaker version of **Conjecture 3**.

**Theorem** (K. 2021+)

A graph $G$ is ternary if and only if $I(H)$ is either contractible or homotopy equivalent to a sphere for every induced subgraph $H$.

# Proof idea

Let $G$ be a graph.
For a vertex $v$, let $N(v) := \{u \in V(G) : uv \in E(G)\}$ and $N[v] := N(v) \cup \{v\}$.

Note that any independent set containing $v$ is contained in $V(G) - N(v)$.

For any vertex $v$,

- $I(G) = I(G - v) \cup I(G - N(v))$,
- $I(G - v) \cap I(G - N(v)) = I(G - N[v])$, and
- $I(G - N(v))$ is contractible.

# Topological tools

**Mayer-Vietoris Sequence**

For simplicial complexes $A$ and $B$, the following sequence is exact.

$$\cdots \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(A \cup B) \to \tilde{H}_{i-1}(A \cap B) \to \cdots .$$

For a graph $G$ and a vertex $v$ of $G$,

- the following sequence is exact:

$$\cdots \to \tilde{H}_i(I(G - N[v])) \to \tilde{H}_i(I(G - v)) \to \tilde{H}_i(I(G)) \to \tilde{H}_{i-1}(I(G - N[v])) \to \cdots ,$$

and

- if $N[v] \neq V(G)$, then $I(G) \simeq I(G - v)/I(G - N[v])$.

# Topological tools

**Lemma 1**

Let $G$ be a graph and $v \in V(G)$ such that $N[v] \neq V(G)$.

If each of $I(G), I(G-v), I(G-N[v])$ is either contractible or homotopy equivalent to a sphere, then one of the following holds:

1. $I(G)$ is contractible and $I(G-v) \simeq I(G-N[v])$,
2. $I(G-N[v])$ is contractible and $I(G) \simeq I(G-v)$, and
3. $I(G-v)$ is contractible and $I(G) \simeq \Sigma I(G-N[v])$,

where $\Sigma \mathbb{S}^d \simeq \mathbb{S}^{d+1}$.

# Proof sketch

For the sake of contradiction, take a minimal counter-example $G$:

- $G$ is a ternary graph,
- $I(G)$ is neither contractible nor homotopy equivalent to a sphere,
- $I(H)$ is contractible or homotopy equivalent to a sphere, $\forall H \lneq G$.

For disjoint vertex subsets $X, Y$ of $G$, let

$$G(X|Y) := G[V(G) - N[X] - Y] \text{ if } X \text{ is independent,}$$

and let

$$d(X|Y) := \begin{cases} d & \text{if } X \text{ is independent and } I(G(X|Y)) \simeq \mathbb{S}^d, \\ * & \text{otherwise.} \end{cases}$$

**Rmk**: If a graph $H$ has no vertex, then we write $I(H) \simeq \mathbb{S}^{-1}$.

# Proof sketch

## Lemma 2

For $X, Y \subseteq V(G)$ s.t. $X \cup Y \neq \emptyset$ and $X \cap Y = \emptyset$ and a vertex $v \notin X \cup Y$, $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\}))$ equals to one of the following:

$$(*, *, *), (k, *, k), (*, k, k), (k+1, k, *)$$

for some integer $k \geq -1$.

# Proof sketch

**Lemma 3**

There is a non-negative integer $k$ s.t. $\forall v \in V(G)$, $d(\emptyset|v) = d(v|\emptyset) = k$.

**(Proof of main theorem)**

By Lemma 3, $\exists k$ s.t. $\forall v \in V(G)$, $d(v|\emptyset) = d(\emptyset|v) = k$.

**Claim**: $\forall u, v \in V(G)$ s.t. $u \neq v$, $d(u, v|\emptyset) = k - 1$.

By Lemma 2,

$$(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (k, *, k) \text{ or } (k, k-1, *),$$

$$(d(\emptyset|u), d(v|u), d(\emptyset|u, v)) = (k, *, k) \text{ or } (k, k-1, *).$$

$\Rightarrow d(v|u) = *$, $(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (k, k-1, *)$.

Since $d(u, v|\emptyset) = k - 1 \neq * \Rightarrow \{u, v\}$ is an independent set.

$\Rightarrow V(G)$ is an independent set $\Rightarrow I(G)$ is contractible (contradiction).

# Open questions

Q. For a ternary graph $G$, when is $I(G)$ contractible?

Q. For a ternary graph $G$, can we compute the dimension of the sphere that is homotopy equivalent to $I(G)$?

Q. Can we find an analogue for hypergraphs?

# Thank you!