A unified approach to hypergraph Stability.

Joint with Dhruv Mubayi and Christian Reiher

- Nondegenerate Turán Problem.
- Types of Stability.
- Known methods. & results.
- Our method and resuts.
- Example.. (cancellative 3-graphs)

$\mathcal{F}$ is a family of r-graphs. H is an r-graph.

Def: • H is $\mathcal{F}$-free if it does not contain any member in $\mathcal{F}$ as a subgraph. (not necessarily induced)

• Turán number

$$ex(n, \mathcal{F}) = \max\{|H| : H \text{ is } \mathcal{F}\text{-free and } |V(H)| = n\}.$$

• Turán density

$$\pi(\mathcal{F}) = \lim_{n \to \infty} \frac{ex(n, \mathcal{F})}{\binom{n}{r}}.$$

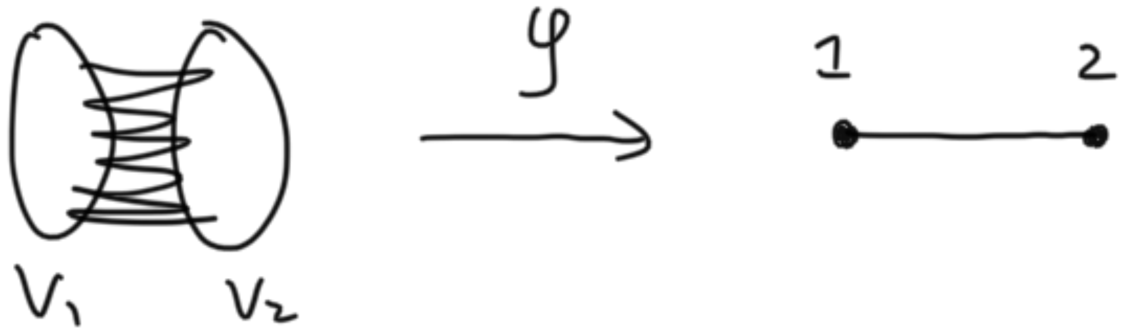
• $\mathcal{F}$ is non-degenerate if $\pi(\mathcal{F}) > 0$

Note: every family considered here is nondegenerate.

Def:

- H is a blowup of G. if $\exists\, \varphi : V(H) \to V(G)$ s.t. $E \in H \iff \varphi(E) \in G$.
- H is G-colorable if H is a subgraph of some blowup of G.

e.g.



$$\varphi(v) = \begin{cases} 1 & \text{if } v \in V_1 \\ 2 & \text{if } v \in V_2 . \end{cases}$$

$$\varphi(v) = \begin{cases} 1 & \text{if } v \in V_1 \\ 2 & \text{if } v \in V_2 \\ 3 & \text{if } v \in V_3 . \end{cases}$$

Rmk: G is K_ℓ-colorable iff G is ℓ-partite.

Thm (Turán 1941)

$$ex(n, K_{\ell+1}) = |T(n,\ell)|.$$

The Turán graph $T(n,\ell)$ = maximum K_ℓ-colorable graph on n vertices

= the balance complete ℓ-partite graph.

Thm (Erdös-Stone, 1946)

$$ex(n, \mathcal{F}) = |T(n, \chi(\mathcal{F})-1)| + o(n^2), \quad \text{where}$$

$$\chi(\mathcal{F}) = \min\{\chi(F) : F \in \mathcal{F}\}.$$

Stability:

Thm (Erdős – Simonovits, 1968)

$$K_{\ell+1} \not\subseteq G \ + \ |G| \approx ex(n, K_{\ell+1}) \implies G \approx T(n, \ell)$$

$\forall \ell \in \mathbb{N}$, $\forall \delta > 0$, $\exists \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$.

If G is an n-vtx $K_{\ell+1}$-free graph with $|G| \geq (1-\varepsilon) \cdot |T(n,\ell)|$.

then G is K_ℓ-colorable after removing at most δn^2 edges.

Thm (Andrásfai – Erdős – Sós 1974)

$$K_{\ell+1} \not\subseteq G. \quad \delta(G) \geq (1 - c_\ell) \cdot \frac{\ell-1}{\ell} \cdot n \implies G \text{ is } K_\ell\text{-colorable.}$$

a positive constant.

Types of stability (of $\mathcal{F}$)

- $\mathcal{F}$ is a family of r-graphs. e.g. $\mathcal{F} = \{K_3\}$
- $\mathcal{G}$ is an infinite family of r-graphs. e.g. $\mathcal{G} = \{$all bipartite graphs$\}$.
- H is an n-vertex r-graph.

(edge stable): $|H| \geq (1-\varepsilon) \cdot ex(n, \mathcal{F}) \Rightarrow$

$H \in \mathcal{G}$ after removing $\leq \delta \cdot n^r$ edges.

(vertex stable): $|H| \geq (1-\varepsilon) \cdot ex(n, \mathcal{F}) \Rightarrow$

$H \in \mathcal{G}$ after removing $\leq \delta \cdot n$ vertices.

(degree stable): $\delta(H) \geq (1-\varepsilon) \cdot \frac{r \cdot ex(n, \mathcal{F})}{n} \Rightarrow H \in \mathcal{G}$.

Obs: degree stable $\Rightarrow$ vertex stable $\Rightarrow$ edge stable.

For graphs :

<u>Def:</u> • A graph F is edge-critical if $\exists e \in F$ s.t. $\chi(F-e) < \chi(F)$.

F is matching-critical if $\exists M \subseteq F$ s.t. $\chi(F-M) < \chi(F)$.

• A family $\mathcal{F}$ is edge-critical / matching-critical if $\exists F \in \mathcal{F}$ s.t. F is edge-critical / matching-critical and $\chi(F) = \chi(\mathcal{F})$.

follows from the AES thm and a theorem of Erdős and Simonovits, 1973.

<u>Thm</u>

$\mathcal{F}$ is degree stable iff $\mathcal{F}$ is edge-critical.

$\mathcal{F}$ is vertex stable iff $\mathcal{F}$ is matching-critical

Probably already well known!

General methods for Stability:

#1 edge-stability. Norin–Yepremyan. 2017. (Local stability.)

#2 Vertex-stability. Pikhurko. 2008. (Zykov symmetrization).

#3. degree-stability. L-Mubayi-Reiher. (Ψ-trick).

Rmk: Every result that can be proved using #2. Can also be proved using #3. Most results that can be proved using #1. Can be proved using #3. ($\exists$ one exception. Frankl-Füredi)

#3 Can strengthen / Simplify results from

cancellative hypergraphs $\begin{cases} \text{Keevash - Mubayi} & 2004 & r=3. \\ \text{Pikhurko.} & 2008. & r=4. \end{cases}$

hypergraph expansions $\begin{cases} \text{Mubayi} & 2006. \\ \text{Brandt - Irwin - Jiang 2017. Norin - Yepremyan 2017.} \\ \text{Hefetz - Keevash. 2013.} \\ \text{Bene Watts - Norin - Yepremyan. 2019} \end{cases}$

Def: • <u>Shadow</u> of H

$$\partial H = \left\{ A \in \binom{V(H)}{r-1} : \exists\, B \in H \text{ s.t. } A \subseteq B \right\}.$$

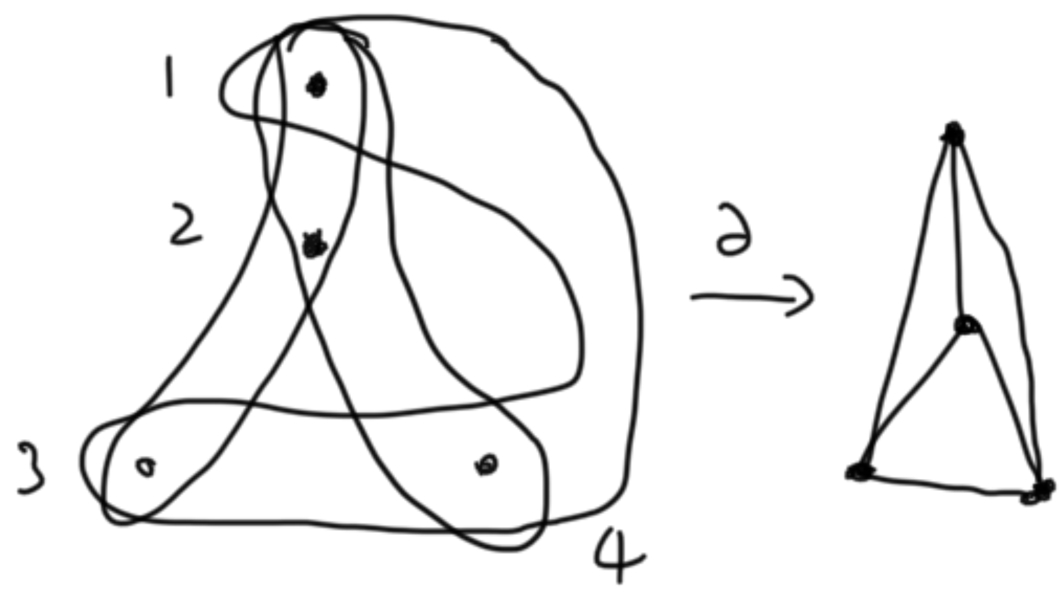
• <u>i-th shadow.</u> $\quad \partial_i H = \partial(\partial_{i-1} H)$.

• In particular. $\partial_{r-2} H$ is a graph.

• $S \subseteq V(H)$ is <u>2-covered</u> if every pair $\{u,v\} \subseteq S$ is contained in an edge.

• H is 2-covered if $V(H)$ is 2-covered.

$\{1,2,3\}$ is 2-covered.



K_4^{3-} is 2-covered.

<u>Obs:</u> $\quad H$ is 2-covered $\iff$ $\partial_{r-2} H$ is complete.

Def:

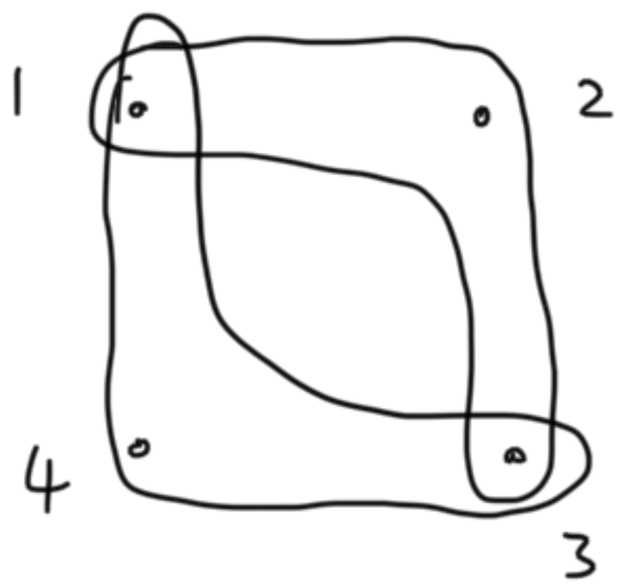
extension of neighbors for graphs.

- The <u>link</u> of v is

$$L_H(v) = \left\{ A \in \binom{V(H)}{r-1} : A \cup \{v\} \in H \right\}.$$

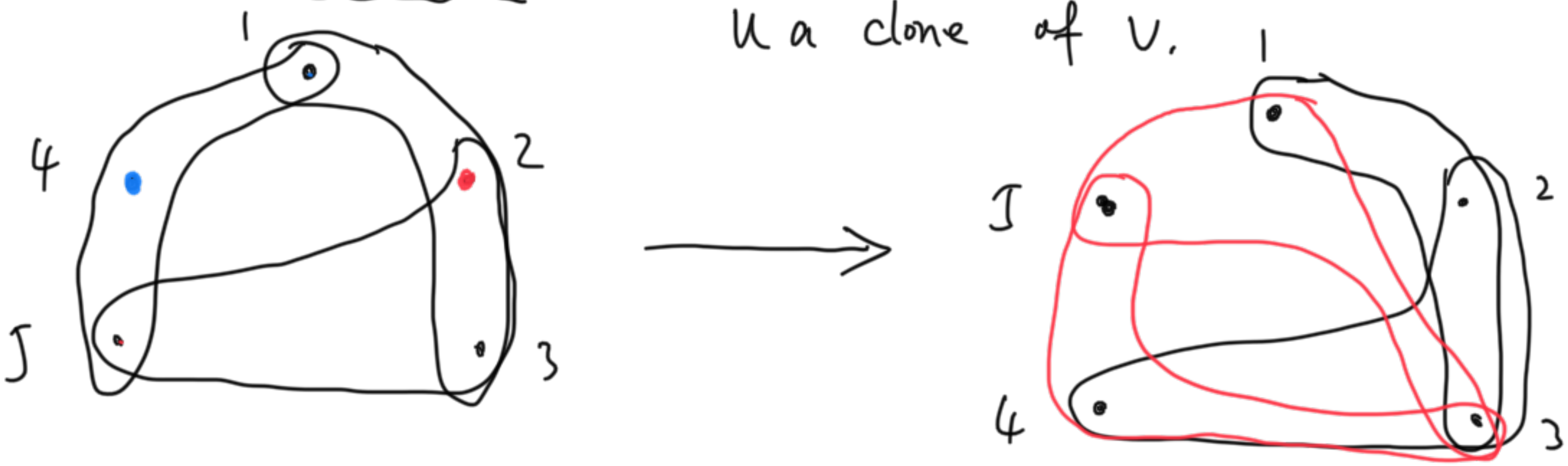
- Two nonadjacent vertices u and v are <u>equivalent</u>, if $L_H(u) = L_H(v)$.

e.g.

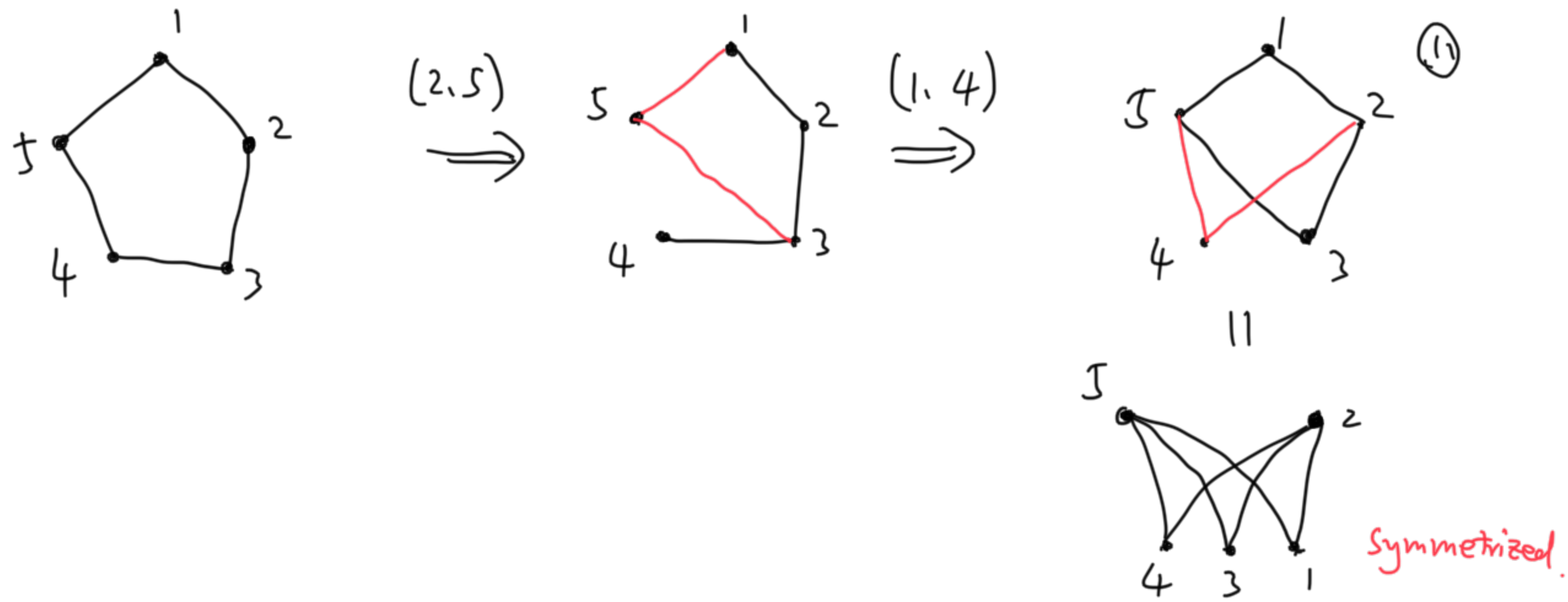


2 and 4 are equivalent.

- <u>Symmetrization</u>. If u, v are not equivalent and $d_H(u) \geq d_H(v)$, then remove all edges containing u and make u a clone of v.



e.g.



Def: H is symmetrized if it does not contain any nonadjacent nonequivalent pair $\{u, v\}$.

Rmk:

- A graph is symmetrized iff it is a blowup of some complete graph. i.e. complete multipartite graphs.
- A hypergraph is symmetrized iff it is a blowup of some 2-covered hypergraph.

$\mathcal{F}$ is a family of r-graphs, $\mathcal{G}$ is an infinite family of r-graphs.

Def: e.g. $\mathcal{F} = \{K_3\}$. $\mathcal{G} = \{$all bipartite graphs$\}$.

① $\boxed{\text{blowup invariant}}$: blowup keeps the $\mathcal{F}$-freeness.

Counter-example C_5

② $\boxed{\text{symmetrized stable}}$ with resp. to $\mathcal{G}$:

every symmetrized $\mathcal{F}$-free r-graph is contained in $\mathcal{G}$.

e.g. every symmetrized K_3-free graph is bipartite.

③ $\boxed{\text{vertex extendable}}$ with resp. to $\mathcal{G}$:

every n-vertex $\mathcal{F}$-free r-graph $\mathcal{H}$ with $\delta(\mathcal{H}) \geq (1-\varepsilon) \cdot \frac{r \cdot ex(n, \mathcal{F})}{n}$

has the following property: $\mathcal{H} - v \in \mathcal{G} \Rightarrow \mathcal{H} \in \mathcal{G}$. $\forall v \in V(\mathcal{H})$.

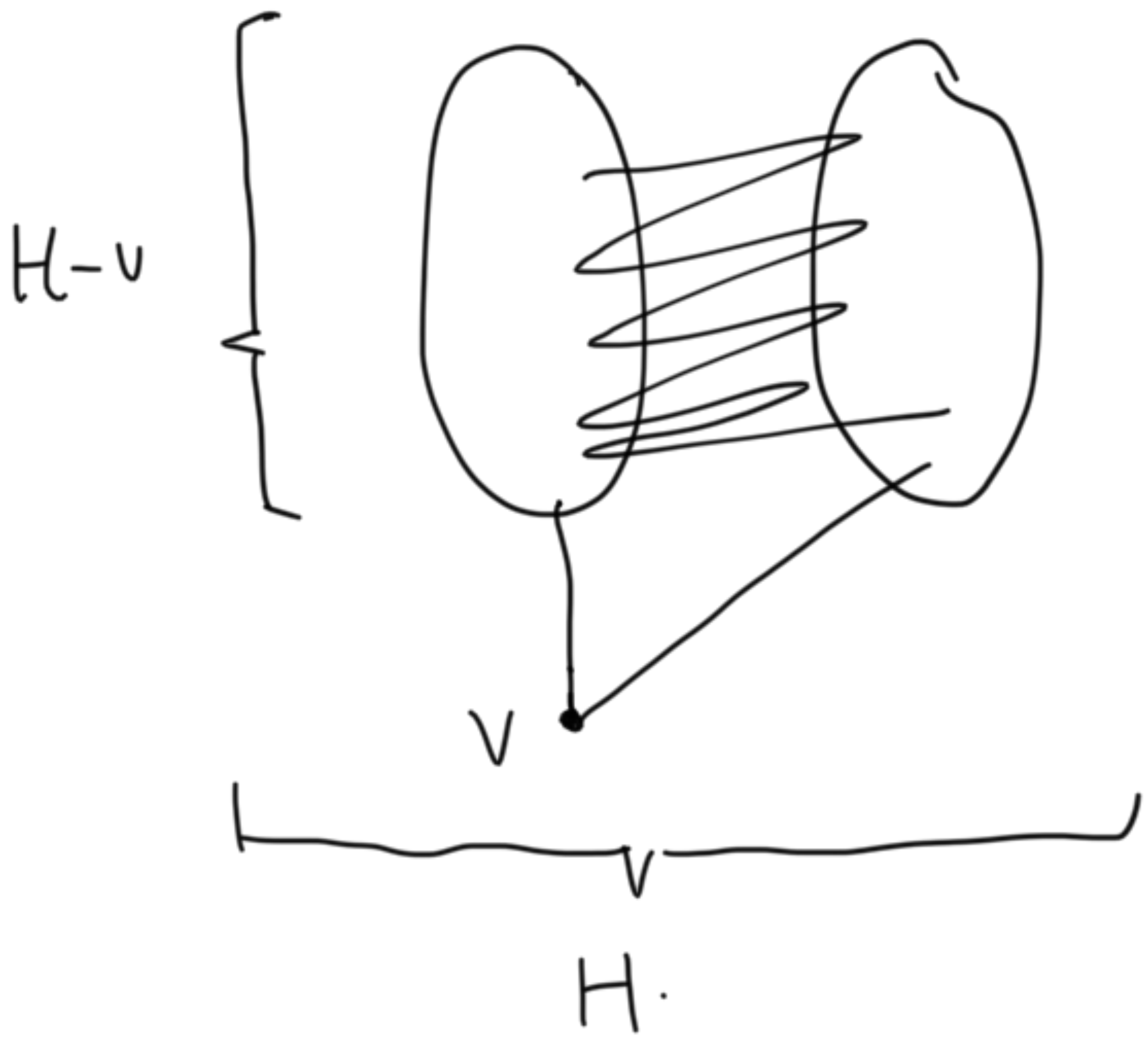
Rmk: In many cases ① & ② are easy to check.

③ is the key and not very hard to check in many cases.

e.g. (K_3 is vertex-extendable with resp. to $\mathcal{G} = \{$ all bipartite graphs $\}$.

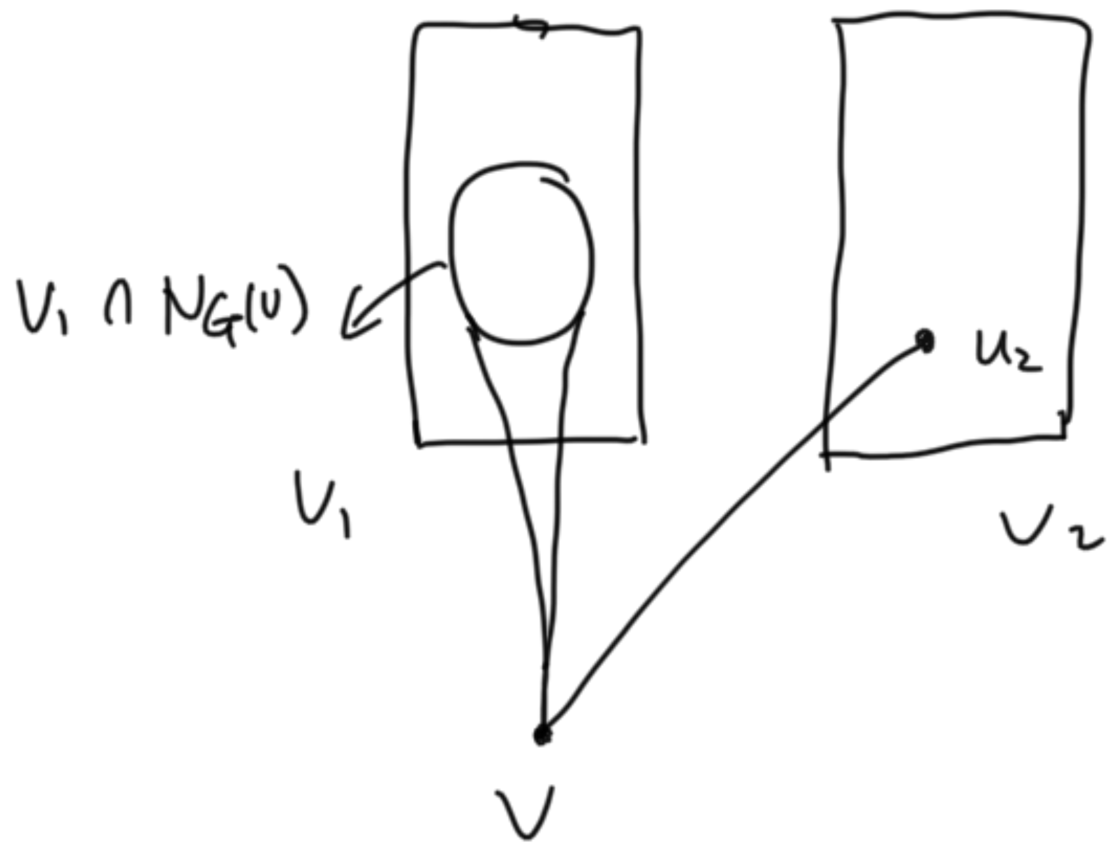
- H is $\triangle$-free
- $\delta(H) \geq (1-\varepsilon) \cdot \frac{n}{2}$.
- $H - v$ is bipartite.

goal $\Rightarrow$ H is bipartite.



$\delta(H) \geq (1-\varepsilon) \cdot \frac{n}{2}$.

<u>Claim:</u> v cannot have neighbors in both V_1 and V_2.



So, H is bipartite.

Thm (L – Mubayi – Reiher, 2021+)

Suppose $\mathcal{F}$ is a nondegenerate family of r-graphs and $\mathcal{G}$ is an (infinite) family of $\mathcal{F}$-free r-graphs. If

1. $\mathcal{F}$ is blowup invariant — i.e. blowup keep $\mathcal{F}$-freeness
2. $\mathcal{F}$ is symmetrized stable with resp. to $\mathcal{G}$. — i.e. $\mathcal{G}$ contains all symmetrized $\mathcal{F}$-free r-graph.
3. $\mathcal{F}$ is vertex extendable with resp to $\mathcal{G}$. — i.e. $H - v \in \mathcal{G} \Rightarrow H \in \mathcal{G}$.

then, every n-vertex $\mathcal{F}$-free r-graph H with $\delta(H) \geq (1-\varepsilon) \cdot \frac{r \cdot ex(n, \mathcal{F})}{n}$.

satisfies $H \in \mathcal{G}$.

e.g. Let $\mathcal{F} = \{K_{\ell+1}\}$ and $\mathcal{G} = \{$all ℓ-partite graphs$\}$.

Apply the theorem above we get the (weaker version) Andrásfai–Erdős–Sós Thm.

Thm (LMR. full version)

Suppose that $\mathcal{F}$ is a nondegenerate family of r-graph.

$\{\mathcal{G}_i\}_{i\in I}$ is a collection of $\mathcal{F}$-free r-graph families.

If ① $\mathcal{F}$ is blowup invariant.

② $\mathcal{F}$ is weakly sym. stable with resp. to $\bigcup_{i\in I} \mathcal{G}_i$.

i.e. $\bigcup_{i\in I} \mathcal{G}_i$ contains all symmetrized $\mathcal{F}$-free r-graph with edge density $\geq (1-\varepsilon)\cdot \pi(\mathcal{F})$

③ $\mathcal{F}$ is vertex extendable with resp. to $\mathcal{G}_i$, $\forall i \in I$.

i.e. $H - v \in \mathcal{G}_i \Rightarrow H \in \mathcal{G}_i$ $\forall i \in I$.

then. every n-vertex $\mathcal{F}$-free r-graph with $\delta(H) \geq (1-\varepsilon) \dfrac{r\cdot ex(n,\mathcal{F})}{n}$. satisfies

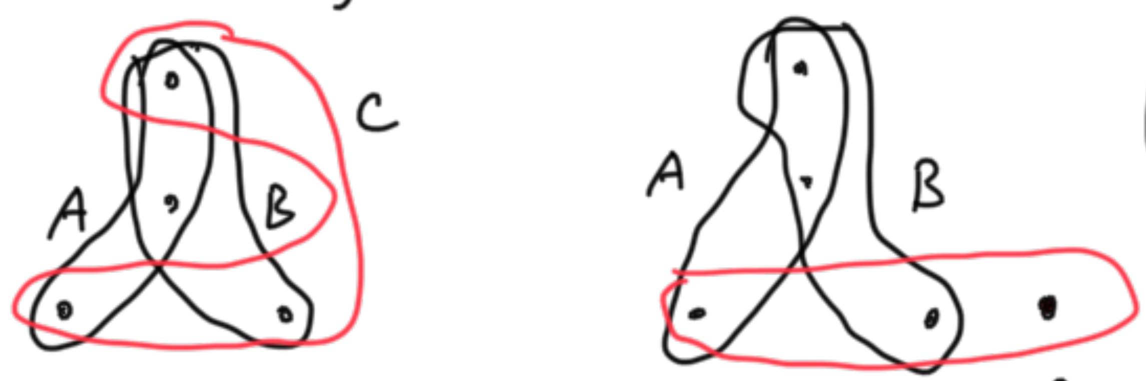
$H \in \mathcal{G}_i$ for some $i \in I$.

Rmk: • Useful for handling families with ≥ 2 extremal configurations.

• $|I|$ is usually the number of "different" extremal configurations

A hypergraph example: (cancellative 3-graph.)

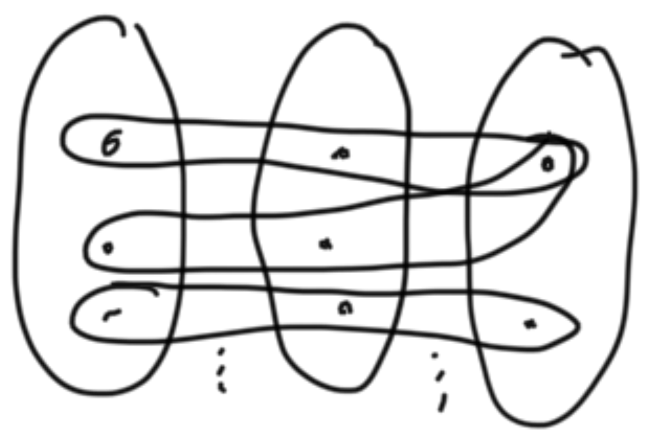
forbidden family $T_3 = \{$



K_4^{3-} $\quad$ F_5 $\}$.

Obs: H is T_3-free iff it does not contain three edges A, B, C with $A \triangle B \subseteq C$.

Thm (Bollobás, 1974) $\quad ex(n, T_3) = |T_3(n,3)| \sim \left(\frac{n}{3}\right)^3$



$T_3(n,3)$

Thm (Keevash - Mubayi 2004) $\quad T_3$ is edge stable.

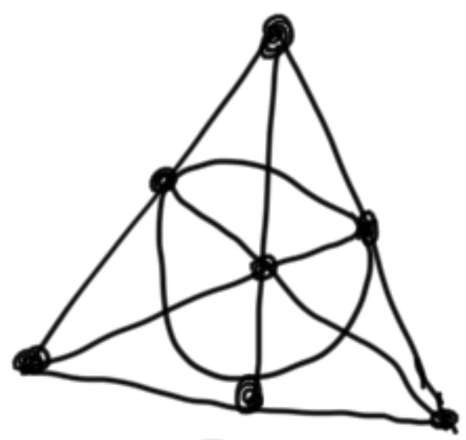
Rmk: Using Pikhurko's method. T_3 is vertex stable.

Thm (L – Mubayi – Reiher 2021+). T_3 is degree stable.

proof sketch: ① T_3 is blowup invariant. (easy to check).

Let $\mathcal{G}$ be the family of 3-graphs that can be colored by a Steiner triple system. (i.e. every pair $\{u,v\}$ is contained in exactly one edge).

K_3^3



Fano Plane.

. . . .

② T_3 is symmetrized stable with resp. to $\mathcal{G}$.

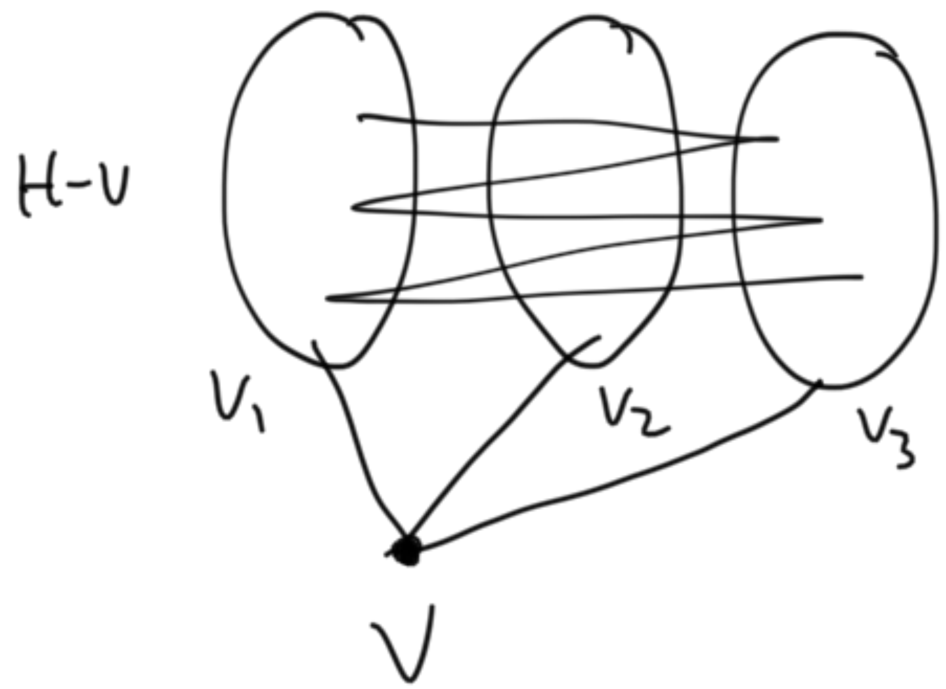
rmk: just need to show that every 2-covered T_3-free 3-graph is a Steiner triple system.

③ T_3 is vertex extendable with resp. to $\mathcal{G}$.

- H is T_3-free
- $\delta(H) \geq (1-\varepsilon)\left(\frac{n}{3}\right)^2$
- $H - v \in \mathcal{G}$.

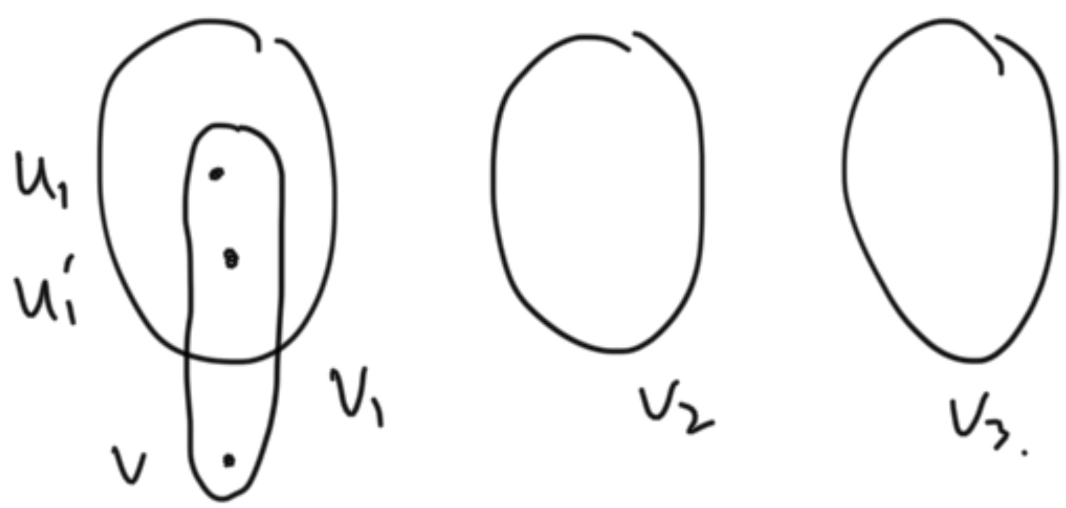
$\xRightarrow{\text{goal}}$ $H \in \mathcal{G}$.

Fact: every m-vertex 3-graph in $\mathcal{G}$ with minimum degree $\geq (1-2\varepsilon)\left(\frac{m}{3}\right)^2$ is 3-partite.



Claim: $L_H(v)$ is a bipartite graph between V_i and V_j for some $i, j \in [3]$.

Pf: Step 1: $L_H(v) \cap \binom{V_i}{2} = \emptyset$ $\forall\, i \in [3]$

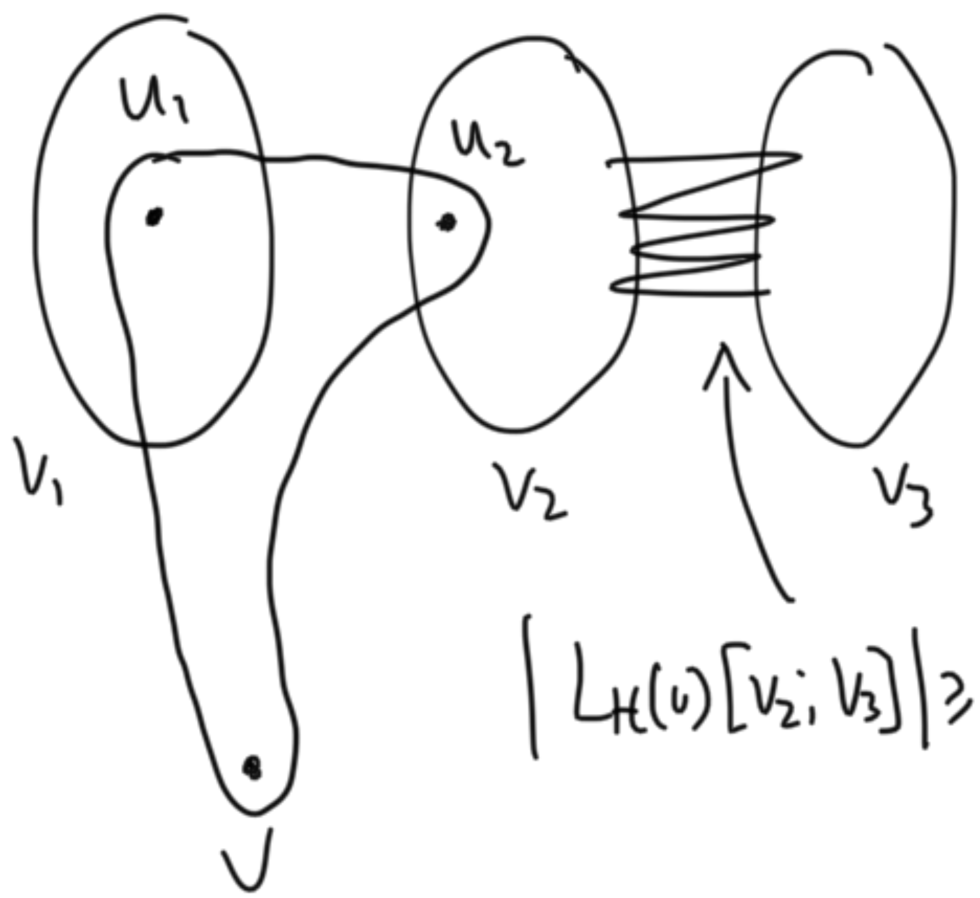


<u>Step 2:</u> By the Pigeonhole Principle, $\exists\ i, j \in [3]$, s.t.

$$\left| L_H(v)[V_i, V_j] \right| \geq \frac{1}{3} \left| L_H(v) \right|.$$

Assume $\{i, j\} = \{2, 3\}$, then.

<u>Claim:</u> $L_H(v)[V_2, V_3] = L_H(v)$



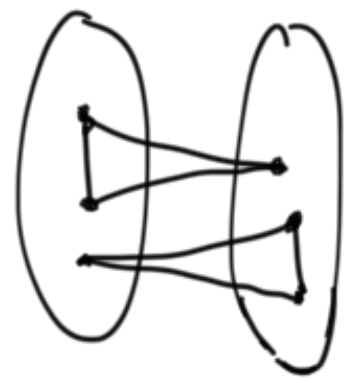
$$\left| L_H(v)[V_2, V_3] \right| \geq \frac{1}{3} \cdot (1-\varepsilon) \cdot \left(\frac{n}{3}\right)^2$$

By Thm, every T_3-free H with $\delta(H) \geq (1-\varepsilon)\left(\frac{n}{3}\right)^2$ is 3-partite.

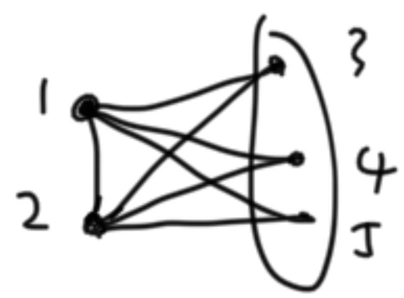
$\vdash [1]$

Rmk : • Apply our method to simplify the proofs for the Fano plane and $\mathbb{F}_{3,2}$?

Fano plane



complete bipartite 3-graph



$\mathbb{F}_{3,2}$.

Semibipartite 3-graph

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}$ Keevash – Sudakov, Füredi – Simonovits. 2005

$\underbrace{\qquad\qquad\qquad\qquad}$ Füredi – Pikhurko – Simonovits. 2005

• Applications in other extremal problems ?

Inducibility, digraphs. ...

Thank You !