On a conjecture of Bondy and Vince

Joint work with Jie Ma

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Overview

All graphs referred here are simple.
Erőds et. al. asked whether every graph with minimum degree at least three contains two cycles whose lengths differ by one or two.

Theorem 1 (Bondy and Vince, 1998)

With the exception of K_1 and K_2, every graph having at most two vertices of degree less than three contains two cycles of lengths differing by one or two.

Introduction

They further conjectured the following generalization.

Conjecture 2 (Bondy and Vince, 1998)

Let k be any nonnegative integer. With finitely many exceptions, every graph having at most k vertices of degree less than three has two cycles whose lengths differ by one or two.

Main result

we confirm the above conjecture of Bondy and Vince by the following.

Theorem 3 (Gao and Ma, 2020)

Every graph, having at most k vertices of degree less than three and at least $5k^2$ vertices, contains two cycles whose lengths differ by one or two.

We say a pair of cycles is *good* if their lengths differ by one or two.

Corollary

Let G be an n-vertex graph with min-degree at least three. Then one can derive that by deleting any $\sqrt{n}/5$ edges from G, the remaining graph still contains a good pair of cycles. Also by repeating the following procedure: first apply this theorem to find a pair of two cycles of lengths differing by one or two and then delete two edges to destroy these two cycles, one can in fact find $\Omega(\sqrt{n})$ such pairs of cycles in G.

Let $\mathcal{B}(G)$ denote the set of all vertices with degree at most two in a graph G.
Let $f(1) = f(2) = 3$, $f(3) = 14$, $f(4) = 56$, $f(5) = 116$ and $f(k) = 5k^2$ for $k \geq 6$.
We will prove by induction on k that every graph G with $|\mathcal{B}(G)| \leq k$ and at least $f(k)$ vertices contains a good pair of cycles.

Claim 1

Let H be a graph with $|\mathcal{B}(H)| = k$, minimum degree $\delta(H) \geq 2$ and no good pair of cycles. If $k = 3$ or $k \geq 4$ and $|V(H)| \geq f(k-1) + f(3)$, then H is 2-connected.

$k=2$ ✓

Assume $k-1$.

Proof of claim 1

- A cut-vertex u in H, Let B_1 be a component of $H - \{u\}$ and $B_2 = H - \{u\} - B_1$.
- For $i \in [2]$, let $b_i = |\mathcal{B}(H) \cap B_i|$, and we have $b_i \geq 2$.
- By induction we have $f(b_1 + 1) + f(b_2 + 1) > |V(H_1)| + |V(H_2)| > |V(H)| \geq f(k-1) + f(3) = \max_{2 \leq s \leq k/2}\{f(s+1) + f(k-s+1)\}$

$b_1 + b_2 \leq k$.

$H_1 = B_1 \cup \{u\}$. $H_2 = B_2 \cup \{u\}$. $|\mathcal{B}(B_i \cup \{u\})| \leq b_i + 1$.

$$f(b_1+1) + f(b_2+1) > \max_{2 \leq s \leq k/2} \{f(s+1) + f(k-s+1)\}.$$

A cycle C in a graph H is called *feasible* if it is induced and $H - C$ is connected.

Claim 2

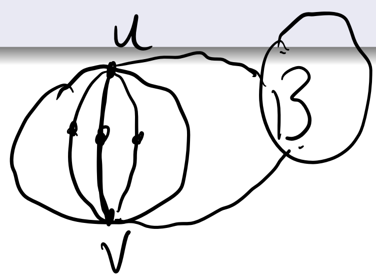
Let H be a 2-connected graph with $|\mathcal{B}(H)| = k$ and no good pair of cycles. If $|V(H)| \geq f(k-1)+1$, then there exists a feasible cycle in H.

A Lemma

Let C be a cycle in a graph G. A *bridge* of C is either a chord of C or a subgraph of G obtained from a component B in $G - V(C)$ by adding all edges between B and C. We call vertices of the bridge not in C *internal*.

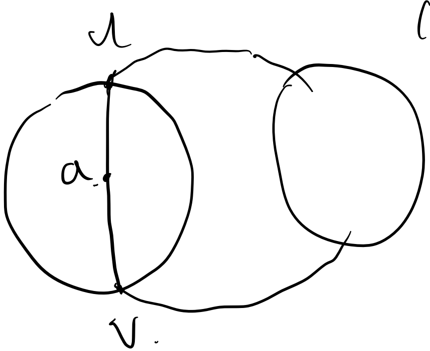
Lemma 4 (Bondy and Vince, 1998)

Let G be a 2-connected graph, not a cycle, and let C be an induced cycle in G some bridge B of which has as many internal vertices as possible. Then either B is the only bridge of C, or else that B is a bridge containing exactly two vertices u, v in $V(C)$ and every other bridge of C is a path from u to v.



Proof of claim 2

- By Lemma 4 we may assume that B is a bridge containing exactly two vertices $u, v \in V(C)$ and there exists another bridge P of C which is a path from u to v.

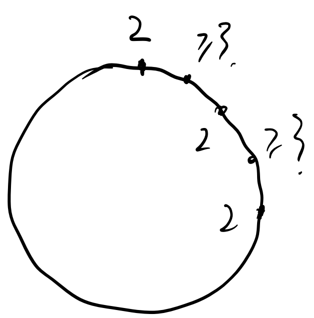


consider H − a.

$B(H-a) \leq k-1$

Claim 3

Let H be a 2-connected graph with $|\mathcal{B}(H)| = k$ and no good pair of cycles, whose order is at least $f(2) + 4$ for $k = 3$ and at least $f(k-1) + f(3) + 2k$ for $k \geq 4$. Let C be any feasible cycle in H and let $A = N_C(H - C)$. Then C has length at most $2k$ and divisible by four, whose vertices alternate between A and $\mathcal{B}(H) \cap V(C)$.



Proof of claim 3

Assume $C = u_0 u_1 \cdots u_{r-2} u_r$.

- There is no pair $(u_i, u_{i+\lfloor r/2 \rfloor+1}) \in A$ where the subscript is modulo r.
- Let H' be obtained from H by deleting $\mathcal{B}(H) \cap V(C)$. Then we know $|A| = |\mathcal{B}(H) \cap V(C)| = |C|/2$ and C is an even cycle of length at most $2k$.

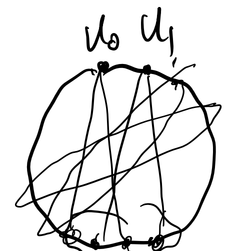
$\lfloor \frac{r}{2} \rfloor + 1$ is odd

- There exist no two consecutive vertices in C belonging to A. $\Rightarrow 4 \mid r$.

H' we know. $|\mathcal{B}(H) \cap V(C)| \le |A| \Leftarrow$ if $|\mathcal{B}(H')| \le k-1$, we are done.

$|A| \le \frac{|C|}{2}$

$\Rightarrow |A| = |\mathcal{B}(H) \cap V(C)| = \frac{|C|}{2}$

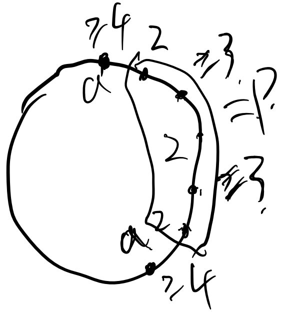


$\Rightarrow |A| \le \frac{|C|}{2} - 1$

Claim 4

Let H, C, A be from Claim 3. Let $D = \{v \in V(C) | \; d_H(v) \leq 3\}$ and $A' = \mathcal{B}(H - D) \setminus \mathcal{B}(H)$. Then $|V(C) \setminus D| \leq 1$, $|A| = |A'| = |C|/2$ with $A \cap A' = V(C) \setminus D$, and every vertex in $A - V(C) \setminus D$ is adjacent to a unique vertex in $A' - V(C) \setminus D$; moreover, $H - D$ is 2-connected with $|\mathcal{B}(H - D)| = |\mathcal{B}(H)|$.

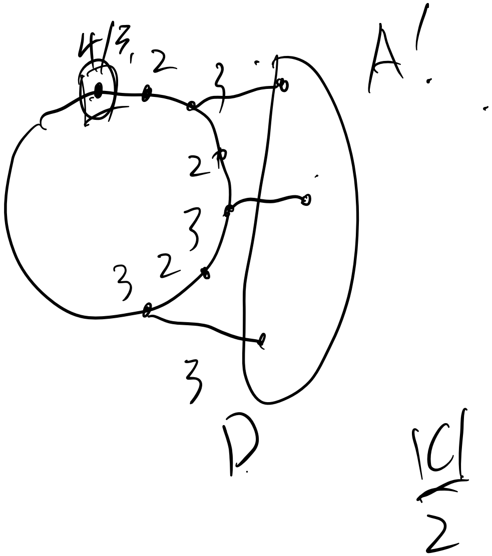
$|D| = |C| - 1$ or $|C|$



consider $H - P$, we know, $\mathcal{B}(H-P) \leq k-1$

So $|V(C) \setminus D| \leq 1$

Proof of claim 4



if $B(H-D) \le k-1$, we are done.

So $B(H-D) = k-1$

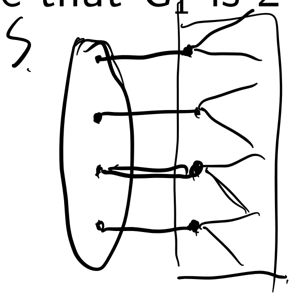
$$|A'| = |A| = \frac{|C|}{2}$$

Proof of Main result

- Let G_0 be a graph with $|\mathcal{B}(G_0)| = k$ and at least $f(k)$ vertices. Suppose for a contradiction that G_0 has no pair of cycles whose lengths differ by one or two.
- Let G_1 be a graph obtained form G_0 by deleting all vertices with degree at most 1.
- Then $|\mathcal{B}(G_1)| = k$ and $|V(G_1)| \geq f(k) - k$, $\delta(G_1) \geq 2$. Since $|V(G_1)| \geq f(k) - k \geq f(k-1) + f(3)$ for $k \geq 4$, by Claim 1 we conclude that G_1 is 2-connected.

G_1

$B(G_1) \leq k$



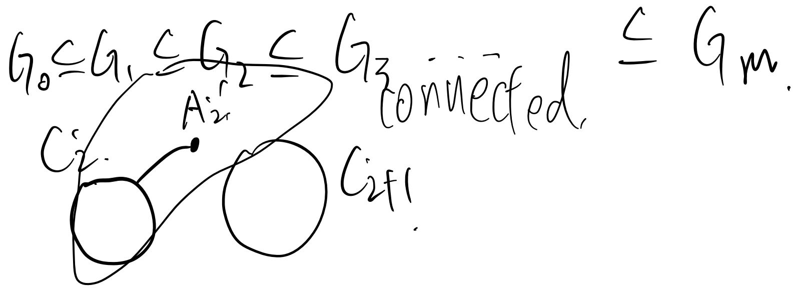
$\Rightarrow \delta(G_1) \geq 2$.

Proof of Main result

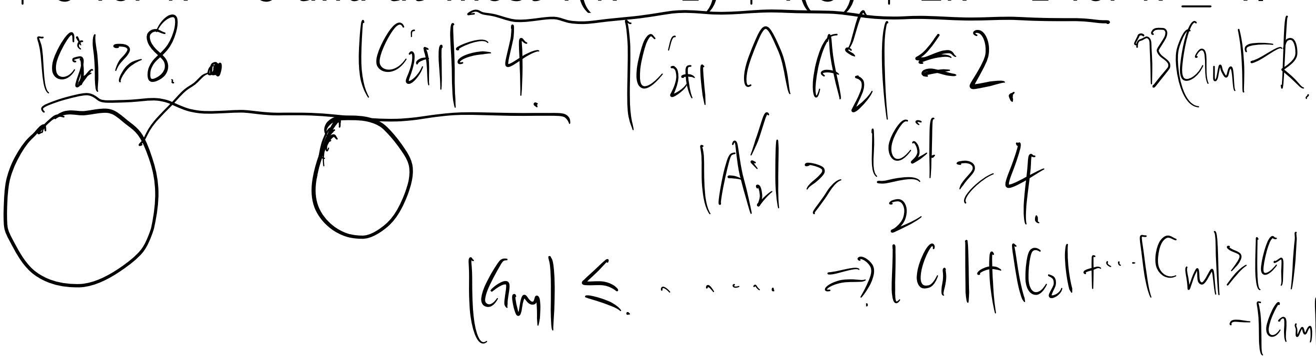
Now suppose we have defined G_i for some $i \geq 1$. If G_i is 2-connected with $|\mathcal{B}(G_i)| = k$ and of order at least $f(2) + 4$ for $k = 3$ and at least $f(k-1) + f(3) + 2k$ for $k \geq 4$, then

- let C_i be a feasible cycle in G_i (by Claim 2), with the preference to be a four-cycle,
- let $A_i = N_{C_i}(G_i - C_i)$, and further
- let $D_i = \{v \in V(C_i) | \ d_{G_i}(v) \leq 3\}$, $G_{i+1} = G_i - D_i$ and $A_i' = \mathcal{B}(G_{i+1}) \setminus \mathcal{B}(G_i)$.

Proof of Main result



- If $A_i' \not\subseteq \mathcal{B}(G_{i+1}) \cap V(C_{i+1})$, then C_{i+1} is a feasible cycle in G_i.
- There exists some t such that $|C_1| = \ldots = |C_t| = 4$ and $|C_i| \geq 8$ for each $t+1 \leq i \leq m-1$.
- The reason we terminate at G_m is because the order of G_m is at most $f(2)+3$ for $k=3$ and at most $f(k-1)+f(3)+2k-1$ for $k \geq 4$.



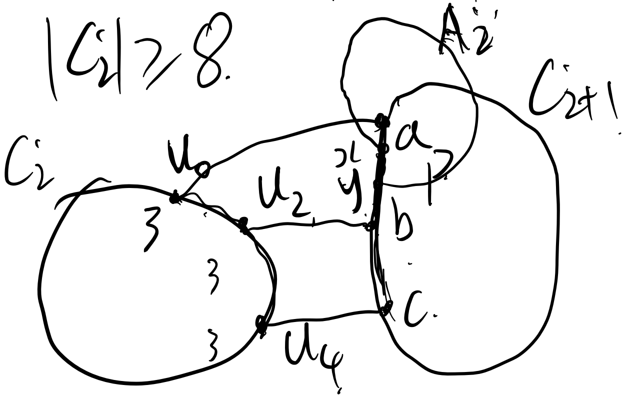
Proof of Main result

So $\Rightarrow f(k) \le |G_0| \le |C_1| + |C_2| + \cdots + |C_{m-1}| + |G_m| + 1k$

$\le 5k + f(k-1)$ contradiction.

- For each $1 \le i \le t-1$, C_{i+1} is feasible in G_i. One can conclude that in fact all four-cycles $C_1, C_2, \ldots, C_t$ are feasible in G_1.
- For each $i \ge t+1$, C_{i+1} is feasible in G_i. So $C_{t+1}, C_{t+2}, \ldots, C_{m-1}$ are feasible in G_{t+1}.

$|C_i| \ge 8$



$|C_1| + |C_2| + \cdots |C_t| \le 2k$

$|C_{t+1}| + \cdots |C_{m-1}| \le 2k$

P be a path in C_{i+1} between a, b

$2 \mid |P|$ if $|P| \equiv 2 \mod 4$

$\Rightarrow 4 \mid |P|$ $|P_{u_0 u_4}| = 4 \Rightarrow P_{ac} \equiv 2 \mod 4$

Problem

We believe that the bound $O(k^2)$ can be further improved (perhaps, to a linear term $O(k)$).

Thank You!