Crux and long cycles in graphs

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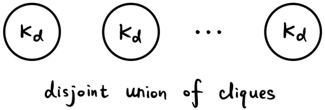
Contents

Fact

Any cyclic graph G contains a cycle of length linear in its average degree, i.e. $\Omega(d(G))$.

- Remove low-degree vertices to obtain a subgraph H with $\delta(H) \geq d(G)/2$.
- Then greedily extend a path to find a cycle in H of length at least $\delta(H) + 1$.

Note that this linear in average degree bound is the best possible.



Motivation 1

Definition (Crux)

For a constant $\alpha \in (0,1)$, a connected subgraph $H \subseteq G$ is an α-crux if $d(H) \geq \alpha \cdot d(G)$. Define the α-crux function, $c_\alpha(G)$, of G to be the order of a minimum α-crux in G, that is,

$$c_\alpha(G) = \min\{|H| : H \subseteq G \text{ and } d(H) \geq \alpha \cdot d(G)\}.$$

- $c_\alpha(G) > \alpha \cdot d(G)$.
- $c_\alpha(G) \geq c_{\alpha'}(G)$ for $\alpha \geq \alpha'$.
- If $H \subseteq G$ with $d(H) \geq d(G)/2$, then $c_{2\alpha}(H) \geq c_\alpha(G)$.

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Motivation 1

We investigate the following ‘replacing average degree by crux’ paradigm.

Question A

Suppose we have a result guaranteeing the existence of a certain substructure whose size is a function of $d(G)$ (or $\delta(G)$). Under what circumstances can we replace $d(G)$ (or $\delta(G)$) with $c_\alpha(G)$?

- Positive instances for the above question would lead to improvements on embedding problems for graph classes whose crux size is much larger than their average degree.

Graphs whose crux size is much larger than their average degree:

- Hypercubes Q^m:

$$c_\alpha(Q^m) \geq 2^{\alpha m}$$

  [Every subgraph G of Q^m with average degree d has at least 2^d vertices.]

- Hamming graphs $H(m,r)$:

$$c_\alpha(H(m,r)) \geq r^{\alpha m}$$

- $K_{s,t}$-free graphs G with $s,t \geq 2$:

$$c_\alpha(G) \geq \frac{(\alpha d(G))^{s/(s-1)}}{2t}$$

  [For every $K_{s,t}$-free graph H with $s,t \geq 2$, we have $2t|H| \geq (d(H))^{s/(s-1)}$. (Kővári, Sós and Turán, 1954)]

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Motivation 1

Let us first see an example of a positive answer to Question A.

Theorem (Komlós and Szemerédi, 1996; Bollobás and Thomason, 1998)

Every graph G contains a topological clique of order $\Omega(\sqrt{d(G)})$.

Theorem (Im, Kim, Kim and Liu)

Every graph G contains a topological clique of order $\Omega(\sqrt{c_\alpha(G)}/(\log c_\alpha(G))^{1/2+o(1)})$.

Conjecture (Mader, 1999)

Every C_4-free graph G contains a topological clique of order $\Omega(d(G))$.

Recall that $c_\alpha(G) = \Omega(d^2(G))$ when G is a C_4-free graph.

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Motivation 2

Theorem (Dirac, 1952)

Every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle.

- Any graph satisfying Dirac's condition is dense, having $\Theta(n^2)$ edges.
- How long a cycle we can ensure in a well-connected **sparse** graph?

Expanders are typically well-connected sparse graphs in which vertex subsets exhibit expansions.

Motivation 2

Expander
- **linear** expander G: $\forall\, X \subseteq V(G)$ with $|X| < \frac{n}{2}$, $|N(X)| \geq c|X|$.
- **sublinear** expander

$c \downarrow \rho(|X|) \sim 1/\log^2|X|$

Theorem (Krivelevich, 2019)

Every linear expander contains a cycle of length linear in its order.

- What about sublinear expanders?
- Note that we **cannot** necessarily find a linear-size cycle, unlike the linear expander case.

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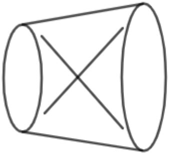

- $K_{n,n/\log^2 n}$ contains a subexpander $H = K_{n/\log^2 n, n/\log^2 n}$ which has a cycle of length linear in the order of H
- Note that H has the average degree about half of $K_{n,n/\log^2 n}$

Is it true that if we cannot find a linear-size cycle in a sublinear expander G, then we can find within G a subgraph H, with about the same average degree as G, that has a cycle of length linear in the order of H?

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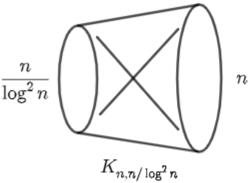

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Crux and cycles

Fact

Any cyclic graph G contains a cycle of length linear in its average degree, i.e. $\Omega(d(G))$.

Theorem (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Let $0 < \alpha < 1$. Then every cyclic graph G contains a cycle of length at least

$$\frac{1-\alpha}{16000} \cdot c_\alpha(G).$$

Sublinear expanders

The following notion of expander was introduced in the 90s by Komlós and Szemerédi.

Definition (Sublinear expander)

A graph G is an (ϵ, t)-*expander* if for any subset $X \subseteq V(G)$ of size $t/2 \leq |X| \leq |V(G)|/2$, we have $|N_G(X)| \geq \rho(|X|) \cdot |X|$, where

$$\rho(x) = \rho(x, \epsilon, t) := \begin{cases} 0 & \text{if } x < t/5, \\ \epsilon / \log^2(15x/t) & \text{if } x \geq t/5. \end{cases} \quad (1)$$

- Note that when $x \geq t/2$, $\rho(x)$ is decreasing, while $\rho(x) \cdot x$ is increasing.

Sublinear expanders

Lemma (Haslegrave, Kim and Liu, 2021)

Let $C > 30, 0 < \epsilon \leq 1/(10C), t > 0, d > 0$ and $\rho(x)$ as in (1). Then every graph G with $d(G) = d$ has a subgraph H such that H is an (ϵ, t)-expander, $d(H) \geq (1 - \delta)d$ and $\delta(H) \geq d(H)/2$, where $\delta := \frac{C\epsilon}{\log 3}$.

Short diameter lemma (Komlós and Szemerédi, 1996)

If G is an n-vertex (ϵ, t)-expander, then for any two vertex sets X_1, X_2 each of size at least $x \geq t/2$, and a vertex set W of size at most $\rho(x)x/4$, there exists a path in $G - W$ between X_1 and X_2 of length at most $\frac{2}{\epsilon}\log^3(\frac{15n}{t})$.

Proof sketch

Theorem (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Let $0 < \alpha < 1$. Then every cyclic graph G contains a cycle of length at least

$$\frac{1-\alpha}{16000} \cdot c_\alpha(G).$$

Corollary (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Let $0 < \alpha < 1$, $0 < \epsilon \le \frac{1-\alpha}{500}$, $t \ge 1$ and suppose $n \ge 150t$. Then every n-vertex (ϵ, t)-expander G contains a cycle of length

$$\max\left\{ \frac{\epsilon}{32} c_\alpha(G) \ , \ \frac{\epsilon n}{1200 \log^2 n} \right\}.$$

Contents

Application 1

Conjecture (Long, 2013)

Any subgraph of the hypercube Q^m that has average degree d contains a path of length at least $2^d - 1$.

- Long gave a weaker bound of a path of length at least $2^{d/2} - 1$.
- The conjecture, if true, would be best possible by considering sub-hypercubes.

Corollary (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Every subgraph of the hypercube with average degree d contains a cycle of length

$$2^{d-o(d)}.$$

Application 2

Conjecture (Long, 2013)

Every subgraph of the discrete torus K_3^m that has average degree at least d contains a path of length at least $3^{d/2} - 1$.

- Long gave a weaker bound of a path of length at least $2^{d/4} - 1$.
- The conjecture, if true, would be best possible by considering sub-torus.

Corollary (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Every subgraph of the Hamming graph $H(m, r)$ with average degree d contains a cycle of length

$$r^{\frac{d}{r-1} - o(d)}.$$

The case $r = 3$ covers the discrete torus.

Contents

Random subgraph

Definition (Random subgraph)

For a given finite graph G and a real $p \in [0, 1]$, let G_p be a random subgraph of G obtained by taking each edge independently with probability p.

- If $G = K_n$, then G_p is simply the Erdős–Rényi binomial random graph $G(n, p)$.
- Analysis of G_p can be used to demonstrate the **robustness** of a graph G with respect to a graph property $\mathcal{P}$.

eg. A robust version of Dirac Theorem:

Theorem (Krivelevich, Lee and Sudakov, 2014)

There exists a positive constant C such that for $p \geq \frac{C \log n}{n}$ and a graph G on n vertices of minimum degree at least $\frac{n}{2}$, w.h.p. the random subgraph G_p is Hamiltonian.

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Long cycles in random subgraphs

Theorem (Frieze, 1986)

For large C, w.h.p. $G(n, C/n)$ has a cycle of length at least $(1-(1-o(1))Ce^{-C})n$.

Theorem (Krivelevich, Lee and Sudakov, 2013)

Given a graph G with minimum degree k, if $pk \to \infty$ as $k \to \infty$, then w.h.p. G_p contains a cycle of length at least $(1-o(1))k$.

- Riordan (2014) subsequently gave a shorter proof.
- Krivelevich and Samotij (2014) later considered graphs without a fixed bipartite subgraph H.
- Ehard and Joos (2018) further improved the error term.

Long cycles in random subgraphs of C_4-free graphs

Theorem (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Let G be a C_4-free graph with minimum degree k. Suppose that $pk \to \infty$ as $k \to \infty$. Then w.h.p. G_p contains a cycle of length at least $(1 - o(1))k^2$.

- Note that the constant 1 is best possible, as there are C_4-free graphs with minimum degree k and order $(1 + o(1))k^2$.
- Recall that $c_\alpha(G) = \Omega(d^2(G)) = \Omega(\delta^2(G))$ when G is a C_4-free graph.

Random subgraphs of the hypercube are also well studied.

Theorem (Haslegrave, H., Kim, Liu, Luan and Wang, 2021+)

Let Q^m be the m-dimensional hypercube. If $p = \frac{1+\epsilon}{m}$, where $\epsilon > 0$, then w.h.p. Q_p^m contains a cycle of length $\frac{2^m}{4m^{32}} = 2^{(1-o(1))m}$.

- Recall that $c_\alpha(Q^m) \geq 2^{\alpha m}$.
- Note that while this paper was being prepared, Erde, Kang and Krivelevich (2021+) proved the above theorem with a better error term $\Omega(\frac{2^m}{m^3 \log^3 m})$.

Open problems

Long path in hypercube (resp. discrete torus):

Conjecture (Long, 2013)

Every subgraph of the hypercube Q^m (resp. discrete torus K_3^m) with average degree d contains a path of length at least $2^d - 1$ (resp. $3^{d/2} - 1$).

Long cycle in random subgraph of hypercube:

Question (Erde, Kang and Krivelevich, 2021+)

Let $\epsilon > 0$ and $p = \frac{1+\epsilon}{m}$. Is it the case that w.h.p. Q_p^m contains a cycle of length $\Omega(2^m)$?

Conjecture (Condon, Díaz, Girão, Kühn and Osthus, 2021)

Suppose that $p = p(m)$ satisfies that $pm \to \infty$ as $m \to \infty$. Then w.h.p. Q_p^m contains a cycle of length $(1 - o(1))2^m$.

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Thanks for your attention!