Orientations of Graphs with Forbidden out-degree Lists and Combinatorics of Eulerian-type polynomials

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Some Notations

Given a graph G and a function $F : V(G) \to 2^{\mathbb{N}}$. If there is an orientation D such that

$$\deg_D^+(v) \notin F(v), \quad \text{for } \forall v \in V(G).$$

Then we call that G has an F-avoiding orientation.

Introduction

In 1976, Frank and Gyárfás proved that for a graph G and two mappings $a, b : V(G) \to \mathbb{N}$ satisfying $a(v) \leq b(v)$ for every vertex v, G has an orientation D satisfying $a(v) \leq \deg_D^+(v) \leq b(v)$ for every vertex v if and only if for each subset $U \subseteq V(G)$,

$$\sum_{v \in U} a(v) - e(U, \overline{U}) \leq |E(G[U])| \leq \sum_{v \in U} b(v),$$

where $e(U, \overline{U})$ is the number of edges joining U and $\overline{U} = V(G) \setminus U$, and $G[U]$ is the subgraph of G induced by U.

Recently, Akbari, Dalirrooyfard, Ehsani, Ozeki and Sherkati considered the similar problem of finding a graph orientation that avoids a certain out-degree at each vertex. And they posed following conjecture.

Conjecture

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If

$$|F(v)| \leq \frac{1}{2}(\deg_G(v) - 1)$$

for each $v \in V(G)$, then G has an F-avoiding orientation.

This conjecture is very natural, since there is no orientation such that $\deg_D^+(v) > \frac{deg_G(v)}{2}$.

Background

Conjecture (Tutte's 3-flow conjecture)

Every graph G with no edge-cut of size 1 or 3 admits a nowhere-zero 3-flow.

It has long been known that Tutte's 3-flow conjecture is equivalent to the statement that every 5-regular graph with no edge-cut of size 1 or 3 has an F-avoiding orientation when $F(v) = \{0, 2, 3, 5\}$ at each vertex v.

Related result

There is some result on this problem. In 2020, Akbari et al. proved that

Theorem

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If

$$|F(v)| \leq \frac{1}{4}(\deg_G(v) - 1)$$

for each $v \in V(G)$, then G has an F-avoiding orientation.

Main tool

One tool we used is the following theorem, called the *Combinatorial Nullstellensatz*, introduced by Alon and Tarsi and further developed as a tool by Alon .

Theorem (Combinatorial Nullstellensatz)

Let K be a field, and let f be a polynomial in the ring $K[x_1, \ldots, x_n]$. Suppose that the degree of f is $t_1 + \cdots + t_n$, where each t_i is nonnegative, and suppose that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of K satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, \ldots, s_n \in S_n$ so that

$$f(s_1, \ldots, s_n) \neq 0.$$

Combinatorial Nullstellensatz Application

In graph theory, the Combinatorial Nullstellensatz is typically used for list coloring problems.

In this case, the polynomial f is called *graph polynomial* of a graph G, defined as follows. An orientation D is fixed on G and the incidence matrix is $M = (m_{ve})_{v \in V(G), e \in E(G)}$. Then, the graph polynomial of G is the polynomial in the ring $K[x_v : v \in V(G)]$ (where K is a field) defined

$$f_D^* = \prod_{e \in E(G)} \left(\sum_{v \in V(G)} m_{ve} x_v \right)$$

. If $\operatorname{coef}(\prod x_v^{\deg_D^+(v)}, f_D^*) \neq 0$, then we call this orientation D an *Alon-Tarsi* orientation of G.

Eulerian Polynomial

One convenient property of the graph polynomial of G is that its coefficients can be determined solely by counting Eulerian orientations on G for a complete explanation), which are defined as follows. Given a graph G, an orientation D of G is called *Eulerian* if $\deg_D^+(v) = \deg_D^-(v)$ for all $v \in V(G)$. A subgraph H of G is called *even* if $|E(H)|$ is even and is called *odd* otherwise.

Eulerian polynomial

There is an graph version to describe the coefficient is non-zero.given an orientation D of G, we let $EE(D)$ and $EO(D)$ denote the number of even and odd subgraphs of G that are Eulerian with respect to D, respectively. If G has an orientation D satisfying

$$EE(D) \neq EO(D),$$

then D is an *Alon-Tarsi orientation*.

Construction of our polynomial

Given a graph G and an orientation D. Let F be the forbidden list, then we consider next polynomial:

$$f_D = \prod_{v \in E(G)} \prod_{f_i \in F(v)} \left(\sum_{e \in V(G)} m_{ve} x_e - f_i(v) \right)$$

.

Since we only focus on the highest degree monomial when we use the Combinatorial Nullstellensatz, we can miss the canstant $f_i(v)$ in each factor.

list orientation Polynomial

Thus our polynimial is that given an orientation D,

$$f_D = \prod_{v \in E(G)} \left(\sum_{e \in V(G)} m_{ve} x_e \right)^{|F(v)|}$$

. recall that the list coloring polynomial is

$$f_D^* = \prod_{e \in E(G)} \left(\sum_{v \in V(G)} m_{ve} x_v \right),$$

Is there any association between this two polynomial?

More on the duality

From the linear algebra, we can prove that there is an duality of coefficient of this two polynomials.

Theorem

If $|F(v)| = \deg_D^+(v)$, *then we have*

$$coef(\prod x_e, f_D) = coef(\prod x_v^{|F(v)|}, f_D^*).$$

We can view the list orientation as a dual version of list coloring in some sense.

Related question

Question

Does every graph G have a spanning subgraph with an Alon-Tarsi orientation in which each vertex $v \in V(G)$ has out-degree at least $\frac{1}{2}(\deg_G(v) - 1)$?

For example, Xuding Zhu proved that for even clique K_{2n} and a perfect matching M, There is an Alon Tarsi orientation D of $K_{2n} - M$ with $\deg_D^+(v) = n - 1$. From the duality, we know the question holds for even complete graph.

Our result

Theorem

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If

$$|F(v)| \leq \frac{1}{3}\deg(v) - 1,$$

then G has an F-avoiding orientation.

And our key idea is from next lemma. We first order the vertices in $V(G)$, and give an orientation from the left vertices to right vertices. We use superscript R, L to represent the degree in right side or left side of the vertex.

Theorem

Suppose that there exists an ordering of $V(G)$ and a spanning subgraph H of G such that for each vertex $v_i \in V(G)$, it holds that

$$|F(v_i)| \leq \deg_G^L(v_i) - 2\deg_H^L(v_i) + \deg_H^R(v_i).$$

Then G has an F-avoiding orientation.

The proof of key theorem

Assume the order be $v_1, \ldots v_n$. Then for each value $1 \le j \le n$, we write

$$f_j = \prod_{i=1}^{j} \left(\sum_{e \in E_G^R(v_i)} x_e - \sum_{e \in E_G^L(v_i)} x_e \right)^{t_i},$$

and we observe that $f = f_n$. Given an edge-set $A \subseteq E(G)$, we say that x^A is in the *support* of f if the monomial x^A has a nonzero coefficient in the expansion of f.

We can prove the following stronger claim by induction on j:

For each $1 \le j \le n$, there exists an edge-set $A_j \subseteq E(G)$ such that

(a) $A_j \in supp(f_j)$,

(b) $A_j \cap E_G^R(v_j) \subseteq E_H^R(v_j)$, *and*

(c) *If $k > j$, then $A_j \cap E_G^L(v_k) \subseteq E_H^L(v_k)$.*

we will work in the quotient ring

$$K[x_{e_1}, \ldots, x_{e_{|E(G)|}}]/\langle x_{e_1}^2, \ldots, x_{e_{|E(G)|}}^2 \rangle,$$

where $\langle x_{e_1}^2, \ldots, x_{e_{|E(G)|}}^2 \rangle$ is the ideal generated by the squares of the variables x_{e_i}.

Probability Method

If we randomly order these vertices, the number of neighbors for each vertex on its left side or right side will satisfy the Chernoff bound.

Lemma (Chernoff Bound)

Let X be a binomially distributed variable with parameters n and p. Let $\mu = np$, and let $0 < \delta \leq 1$. Then,

$$\Pr(X < (1+\delta)\mu) \leq \exp\left(-\frac{1}{3}\delta^2\mu\right)$$

and

$$\Pr(X > (1-\delta)\mu) \leq \exp\left(-\frac{1}{2}\delta^2\mu\right).$$

But we need consider all vertices neighbors, so there are some events which are not independent. For example, if $u \in N(N(v))$, then the $\deg_G^R(v)$ and $\deg_G^L(v)$ are not independent. So we have following lemma to deal with the bad events.

Lemma (symmetric form of the Lovász Local Lemma.)

Let $\mathcal{A}$ be a collection of (bad) events in a probability space. Suppose that each bad event in $\mathcal{A}$ occurs with probability at most p and depends on fewer than D other bad events in $\mathcal{A}$. If

$$Dp \leq 1/e,$$

then with positive probability, no bad event in $\mathcal{A}$ occurs.

sub-exponential regular result

Theorem

Let G be a graph of minimum degree δ and maximum degree $\Delta = e^{o(\delta)}$, and let $F : V(G) \to 2^{\mathbb{N}}$. If

$$|F(v)| \leq \left(\sqrt{2} - 1 - o(1)\right) \deg_G(v)$$

for each vertex $v \in V(G)$, then G has an F-avoiding orientation.

The proof detail

recall that

$$f_j = f_{j-1}\left(\sum_{e\in E_G^R(v_j)} y_e - \sum_{e\in E_G^L(v_j)} y_e\right)^{t_j}.$$

Expanding, we see that

$$f_j = f_{j-1}\sum_{a=0}^{t_j}\binom{t_j}{a}(-1)^{t_j-a}\left(\sum_{e\in E_G^L(v_j)} y_e\right)^{t_j-a}\left(\sum_{e\in E_G^R(v_j)} y_e\right)^{a}. \quad (1)$$

The proof detail

We will restrict our attention to terms in the expansion of (1) that occur when $a = m := \min\{t_j, \deg_H^R(v_j)\}$ in the sum. Hence, we only consider the expansion of

$$f_{j-1}\left(\sum_{e\in E_G^L(v_j)} y_e\right)^{t_j-m}\left(\sum_{e\in E_G^R(v_j)} y_e\right)^{m}. \tag{2}$$

We fix an set $A' \subseteq E_H^R(v_j)$ with m edges and observe that $A' \in \mathrm{supp}\left(\sum_{E_G^R(v_j)} y_e\right)^m$. Furthermore, we write $B = A_{j-1} \setminus E_G^L(v_j)$. And we write

$$f_{j-1} = x^B g + r,$$

We will write $E^L = E^L_G(v_j)$ and $b = d + t_j - \deg^R_H(v_j)$. Now, we would like to show that

$$g\left(\sum_{e\in E^L_G(v_j)} y_e\right)^{t_j-m} \neq 0, \qquad (3)$$

in the quotient ring. We also expand g as

$$g = \sum_{\substack{D\subseteq E^L\\ |D|=d}} c_D x^D.$$

Then,

$$g\left(\sum_{e\in E^L} y_e\right)^{t_j-\deg^R_H(v_j)} = (t_j - \deg^R_H(v_j))! \sum_{\substack{Y\subseteq E^L\\ |Y|=b}} \left(\sum_{\substack{D\subseteq Y\\ |D|=d}} c_D\right) x^Y$$