On the kissing numbers of ℓ_p-spheres in high dimensions

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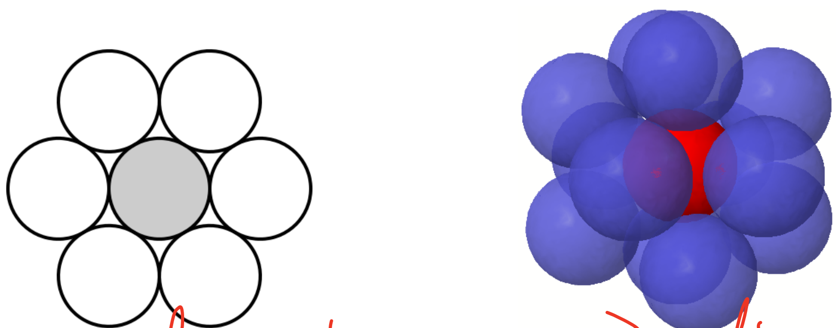
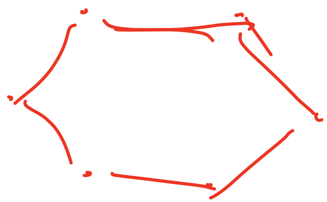
The kissing number problem

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. The (translative) kissing number problem asks the maximum number of nonoverlapping translates $S^{n-1} + \boldsymbol{x}$ that can touch S^{n-1} at its boundary. It is an old and difficult problem in discrete geometry.



2 dimension   3 dimension

Figure: Optimal kissing configurations in 2 and 3 dimensions



icosahedron

正二十面体

Exact answer for kissing numbers

The exact answer is only known in dimensions

- $n = 1$; 2
- $n = 2$; 6
- $n = 3$ (Schütte and van der Waerden, *Math. Ann.*, 1952); 12
- $n = 4$ (Musin, *Ann. of Math.*, 2008); 24
- $n = 8$ and $n = 24$ (Levenšteĭn, *Soviet Math. Dokl.*, 1979, and independently Odlyzko and Sloane, *J. Combin. Theory Ser. A*, 1979).

240 196560 a linear programming method

The best known bounds for kissing numbers

不一样

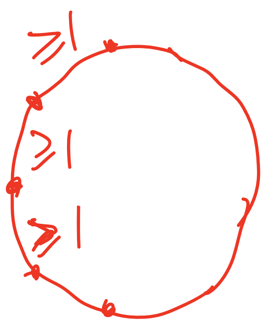
Let $K_2(n)$ be the kissing number of S^{n-1}.

$\approx 1.3\ldots^n$

The best known upper bound is $K_2(n) \le 2^{0.401n(1+o(1))}$ (Kabatjanskiĭ and Levenšteĭn, *Problemy Peredači Informacii*, 1978).

linear programming $\approx 1.15\ldots^n$

The best known lower bound is $K_2(n) \ge cn^{3/2}(2/\sqrt{3})^n$ (Jenssen et al., *Adv. Math.*, 2018). See also Fernández et al. (arXiv:2111.01255) for constant factor improvement. $\approx n^{\frac{1}{2}} \frac{2^n}{(\sqrt{3})^n}$



$\frac{1}{2}$ cap

$$\frac{s(S^{n-1})}{s(\text{半径为1的cap的面积})}$$

Bounds for the kissing numbers of ℓ_p-spheres

kissing number

One can also consider the ~~packing~~ problem of other convex bodies.
For instance, the ℓ_p-spheres (ℓ_p-balls):

$$S_p^{n-1}(R) := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} = R \right\}.$$

We simply write $S_p^{n-1} = S_p^{n-1}(1)$, and let $K_p(n)$ be the kissing number of S_p^{n-1}.

$r \leq 3$

The best known upper bound:

- $1 \leq p \leq 2$, $K_p(n) \leq 3^n - 1$, the Minkowski-Hadwiger theorem, *Arch. Math.*, 1957;
- $p \geq 2$, due to Sah et al., *Adv. Math.*, 2020. implicit

The best known lower bound is due to Xu, *Discrete Comput. Geom.*, 2007.

Larman & Zong

Xie and Ge.

Our result

We improve the results of Xu.
Since our result does not have an explicit formula, we list some numerical results here:

$$K_1(n) \geq \underbrace{2^{0.1247n(1+o(1))}}_{=Xu} + 2^{0.1825n(1+o(1))} + 2^{0.1554n(1+o(1))} + \cdots ;$$

$$K_2(n) \geq \boxed{2^{0.2059n(1+o(1))}} + 2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots ;$$

$$K_3(n) \geq cn\underbrace{\boxed{2^{0.4564n(1+o(1))}}}_{=Xu} + 2^{0.1562n(1+o(1))} + 2^{0.0425n(1+o(1))} + \cdots .$$

Remark 1

In the lower bound for $K_2(n)$, the $2^{0.2059n(1+o(1))}$ term is the same as the lower bound due to Xu, so we improve the lower bound by adding the remainder terms $2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots$.
In the lower bound for $K_3(n)$, the $2^{0.4564n(1+o(1))}$ term is the same as the lower bound due to Xu, so we improve the leading term by a factor of n and add some remainder terms.

Ideas

Our idea comes from coding theory.
The translative kissing number $K_p(n)$ is equal to the largest size of an ℓ_p-*spherical code* with minimum distance 1 (see Lemma 2 below). We choose a discrete set X from S_p^{n-1}. Applying ideas from coding theory, we are able to find a large subset of X, in which points have pairwise distance larger than or equal to 1. This gives a lower bound for $K_p(n)$.

Spherical codes

Let $A_p(n, d)$ be the maximum size of a subset of S_p^{n-1} in which the points have pairwise ℓ_p-distance at least $2d$; that is,

$$A_p(n, d) := \max\{|C| : C \subseteq S_p^{n-1} \text{ and } d_p(\boldsymbol{x}, \boldsymbol{y}) \geq 2d, \forall \boldsymbol{x}, \boldsymbol{y} \in C\},$$

where $d_p(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|_p$ is the ℓ_p-distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. In other words, $A_p(n, d)$ is the largest size of an ℓ_p-spherical code with minimum distance $2d$.

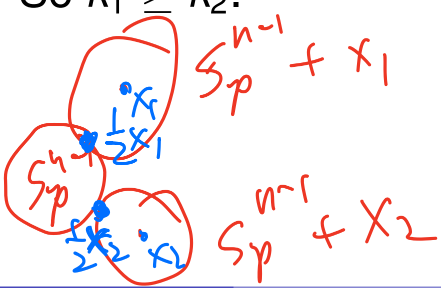
The following lemma is an easy observation.

Lemma 2

$K_p(n) = A_p(n, \frac{1}{2})$.

The translative kissing number $K_p(n)$ of S_p^{n-1} is equal to $A_p(n, 1/2)$.

Sketch of proof:
For convenience, let $k_1 = K_p(n)$ and $k_2 = A_p(n, 1/2)$.
If $S_p^{n-1}, S_p^{n-1} + \boldsymbol{x}_1, S_p^{n-1} + \boldsymbol{x}_2, \ldots, S_p^{n-1} + \boldsymbol{x}_{k_1}$ form a kissing configuration, then $\{\frac{1}{2}\boldsymbol{x}_1, \frac{1}{2}\boldsymbol{x}_2, \ldots, \frac{1}{2}\boldsymbol{x}_{k_1}\}$ is an ℓ_p-spherical code with minimum distance 1, i.e. $k_2 \geq k_1$.
Conversely, if $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{k_2}\}$ is an ℓ_p-spherical code with minimum distance 1, then $S_p^{n-1}, S_p^{n-1} + 2\boldsymbol{x}_1, S_p^{n-1} + 2\boldsymbol{x}_2, \ldots, S_p^{n-1} + 2\boldsymbol{x}_{k_2}$ form a kissing configuration. So $k_1 \geq k_2$.



Constructions

For a positive integer $m \leq n$, which will be determined later, we define a family $\mathcal{J}(m,n)$ of subsets of $\mathbb{R}^n$ recursively. Define $m_1 := m$ and

$$J_1(m,n) := \left\{ \boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \{0, \pm 1\}^n : \sum_{i=1}^{n} |u_i|^p = m \right\}.$$

Suppose we have defined m_i and $J_i(m,n)$. Then we define

$$m_{i+1} := \lfloor m_i / 2^p \rfloor \tag{1}$$

and

$$J_{i+1}(m,n) := \left\{ \boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \{0, \pm (m/m_{i+1})^{1/p}\}^n : \sum_{i=1}^{n} |u_i|^p = m \right\}.$$

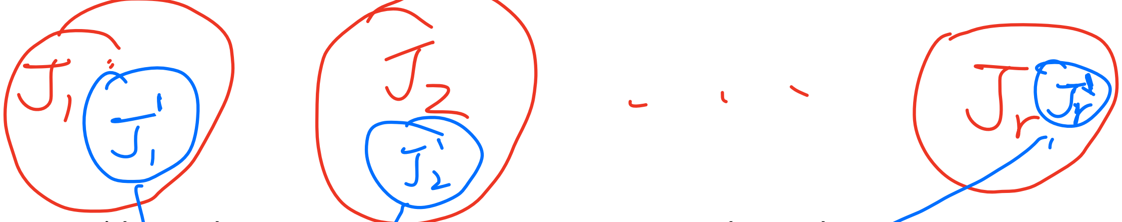
This process terminates when $m_r < 2^p$ for some r ($r = \lfloor \log_{2^p} m \rfloor + 1$ or $\lfloor \log_{2^p} m \rfloor$). So we obtain $\{m_1 > m_2 > \ldots > m_r\}$ and $\mathcal{J}(m,n) = \{J_1(m,n), J_2(m,n), \ldots, J_r(m,n)\}$.

The following proposition is easy to verify.

Proposition 3

For $\mathcal{J}(m,n)$ defined above, the following statements hold.

1. *If $i \neq j$, then $J_i(m,n) \cap J_j(m,n) = \emptyset$.*
2. *For every $1 \leq i \leq r$ and for every $\boldsymbol{u} \in J_i(m,n)$, $\boldsymbol{u}$ has exactly $n - m_i$ zero coordinates.*
3. *For every $1 \leq i \leq r$,*
$$|J_i(m,n)| = \binom{n}{m_i} 2^{m_i}. \tag{2}$$
4. *For every $1 \leq i \leq r$ and for every $\boldsymbol{u} \in J_i(m,n)$, the ℓ_p-norm of $\boldsymbol{u}$ is $m^{1/p}$.*
5. *If $i \neq j$, then for every $\boldsymbol{u} \in J_i(m,n)$ and $\boldsymbol{v} \in J_j(m,n)$, $d_p(\boldsymbol{u},\boldsymbol{v}) \geq m^{1/p}$.*



For every i, let $J_i'(m,n)$ be a largest subset of $J_i(m,n)$ with the property that $d_p(\boldsymbol{u},\boldsymbol{v}) \geq m^{1/p}$ for every $\boldsymbol{u},\boldsymbol{v} \in J_i'(m,n)$. Since we have proved that $d_p(\boldsymbol{u},\boldsymbol{v}) \geq m^{1/p}$ if $\boldsymbol{u} \in J_i'(m,n) \subseteq J_i(m,n)$ and $\boldsymbol{v} \in J_j'(m,n) \subseteq J_j(m,n)$ for $i \neq j$, the set

$$\frac{1}{m^{1/p}} \bigcup_{i=1}^{r} J_i'(m,n) := \left\{ \boldsymbol{x} \in \mathbb{R}^n : m^{1/p}\boldsymbol{x} \in \bigcup_{i=1}^{r} J_i'(m,n) \right\}$$

is an ℓ_p-spherical code with minimum distance 1. So

$$A_p(n,1/2) \geq \left| \frac{1}{m^{1/p}} \bigcup_{i=1}^{r} J_i'(m,n) \right| = \left| \bigcup_{i=1}^{r} J_i'(m,n) \right| = \sum_{i=1}^{r} \left| J_i'(m,n) \right|. \tag{3}$$

A Gilbert-Varshamov type bound

For $1 \leq i \leq r$ and $\boldsymbol{u} \in J_i(m,n)$, define

$$B_{i,n}(\boldsymbol{u}, m) := \left\{ \boldsymbol{v} \in J_i(m,n) : d_p(\boldsymbol{u}, \boldsymbol{v}) < m^{1/p} \right\}.$$

Note that the size of $B_{i,n}(\boldsymbol{u}, m)$ is independent of $\boldsymbol{u}$. If we write $B_{i,n}(m)$ for the size of $B_{i,n}(\boldsymbol{u}, m)$, then

$$B_{i,n}(m) = \sum_{2t+2^p x < m_i} \binom{m_i}{t} \binom{n-m_i}{t} \binom{m_i - t}{x} 2^t. \tag{4}$$

the size

Using the above notations, we have the following theorem.

Theorem 4

For every $1 \leq i \leq r$, we have

$$|J_i'(m,n)| \geq \left\lceil \frac{|J_i(m,n)|}{B_{i,n}(m)} \right\rceil = \left\lceil \frac{\binom{n}{m_i} 2^{m_i}}{B_{i,n}(m)} \right\rceil. \tag{5}$$

Main result

The following corollary is immediate and it is our main result.

Corollary 5

多项!!

$$A_p(n, 1/2) \geq \max_{1\leq m\leq n} \sum_{i=1}^{r} \left\lceil \frac{\binom{n}{m_i} 2^{m_i}}{B_{i,n}(m)} \right\rceil . \tag{6}$$

Remark 6

In the previous result due to Xu (2007), the lower bound for $A_p(n, 1/2)$ is given by $\max_{1\leq m\leq n} \left\lceil \frac{\binom{n}{m_1} 2^{m_1}}{B_{1,n}(m)} \right\rceil$. *So Corollary 5 gives an improvement.*

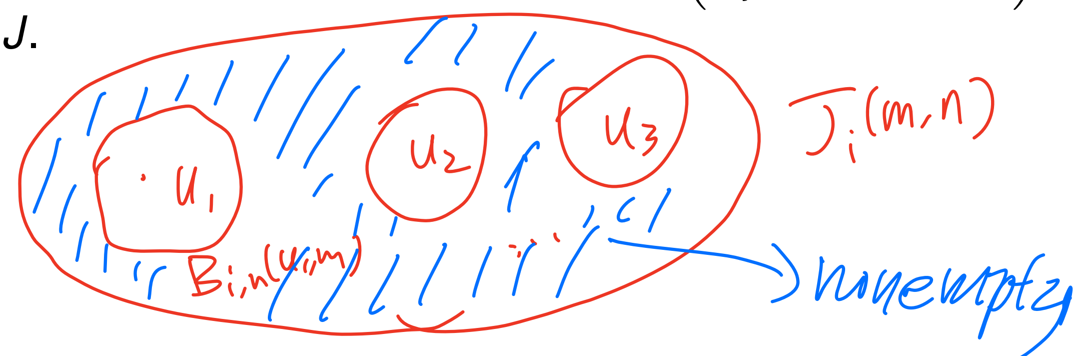
$i=1$

Sketch of proof of Theorem 4:

Let i be given and $J = \left\lceil \frac{|J_i(m,n)|}{B_{i,n}(m)} \right\rceil$. We choose points from $J_i(m,n)$ recursively. At first, we arbitrarily choose $\boldsymbol{u}_1$ in $J_i(m,n)$. If we have chosen $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ for some $k < J$, then the set

$$J_i(m,n) \setminus \left(\bigcup_{j=1}^{k} B_{i,n}(\boldsymbol{u}_j, m) \right)$$

is nonempty and we can choose $\boldsymbol{u}_{k+1}$ from $J_i(m,n) \setminus \left(\bigcup_{j=1}^{k} B_{i,n}(\boldsymbol{u}_j, m) \right)$. And hence $|J_i'(m,n)| \geq J$.



Some numerical results for small p

It seems that there does not exist an explicit formula for the lower bound in Corollary 5. So we give some numerical results.
Define

$$F_p(\sigma) = \frac{\binom{n}{\lfloor \sigma n \rfloor} 2^{\lfloor \sigma n \rfloor}}{\sum_{2t+2^p x < \lfloor \sigma n \rfloor} \binom{\lfloor \sigma n \rfloor}{t} \binom{n - \lfloor \sigma n \rfloor}{t} \binom{\lfloor \sigma n \rfloor - t}{x} 2^t}, \sigma \in (0,1).$$

$$= \frac{|J_i(m,n)|}{B_{i,u}(m)},$$

$$\sigma = \frac{m}{n}.$$

Then by equations (1)-(4) and inequality (6), we have

$$A_p(n, 1/2) \geq \max_{0<\sigma<1} \sum_{i=1}^{r} F_p\left(\frac{\sigma}{2^{(i-1)p}}\right).$$

The behavior of $F_p(\sigma)$

Let $H(\sigma)$ be the entropy function defined as

$$H(\sigma) = \begin{cases} 0, & \text{if } \sigma = 0 \text{ or } \sigma = 1; \\ -\sigma \log_2 \sigma - (1-\sigma)\log_2(1-\sigma), & \text{if } 0 < \sigma < 1. \end{cases}$$

We have the following theorem.

Theorem 7 (Xu, *Discrete Comput. Geom.*, 2007)

We have

$$\lim_{n\to\infty} \frac{1}{n} \log_2 F_p(\sigma) \geq \min_{0 \leq y \leq \min\{\sigma/2, 1-\sigma\}} f_p(\sigma, y),$$

where

$$f_p(\sigma, y) = (\sigma - y)\left(1 - H\left(\frac{\sigma - 2y}{2^p(\sigma - y)}\right)\right) + H(\sigma) - \sigma H\left(\frac{y}{\sigma}\right) - (1-\sigma) H\left(\frac{y}{1-\sigma}\right).$$

Let $g_p(\sigma) = \min_{0 \leq y \leq \min\{\sigma/2, 1-\sigma\}} f_p(\sigma, y)$. We list some numerical results for special values of p.

For $p = 1$, $g_1(\sigma)$ attains its maximum 0.1825 at $\sigma_0 = 0.2605$. So

$$
\begin{aligned}
A_1(n, 1/2) &\geq \max_{0 \leq \sigma \leq 1} \sum_{i=1}^{r} F_1\left(\frac{\sigma}{2^{i-1}}\right) \\
&\geq \sum_{i=1}^{r} F_1\left(\frac{2\sigma_0}{2^{i-1}}\right) \\
&\geq F_1(2\sigma_0) + F_1(\sigma_0) + F_1\left(\frac{\sigma_0}{2}\right) + \cdots \\
&\geq 2^{g_1(2\sigma_0)\cdot n(1+o(1))} + 2^{g_1(\sigma_0)\cdot n(1+o(1))} + 2^{g_1(\sigma_0/2)\cdot n(1+o(1))} + \cdots \\
&= 2^{0.1247n(1+o(1))} + 2^{0.1825n(1+o(1))} + 2^{0.1554n(1+o(1))} + \cdots .
\end{aligned}
$$

$\sigma_0 < 0.5$

add $F_1(2\sigma_0)$ for a better bound.

Remark 8

Although $2^{0.1247n(1+o(1))} + 2^{0.1554n(1+o(1))} + \cdots = o(2^{0.1825n(1+o(1))})$, *we still write them explicitly since they improve the previous bound.*
Talata (Combinatorica, 2000) obtained $A_1(n, 1/2) \geq 2^{0.1825n(1+o(1))}$ *as well.*

For $p = 2$, $g_2(\sigma)$ attains its maximum 0.2059 at $\sigma_0 = 0.3881$. So

$$
\begin{aligned}
A_2(n, 1/2) &\geq \max_{0 \leq \sigma \leq 1} \sum_{i=1}^{r} F_2\left(\frac{\sigma}{2^{2(i-1)}}\right) \\
&\geq \sum_{i=1}^{r} F_2\left(\frac{\sigma_0}{4^{i-1}}\right) \\
&\geq F_2\left(\sigma_0\right) + F_2\left(\frac{\sigma_0}{4}\right) + F_2\left(\frac{\sigma_0}{4^2}\right) + \cdots \\
&\geq 2^{g_2(\sigma_0)\cdot n(1+o(1))} + 2^{g_2(\sigma_0/4)\cdot n(1+o(1))} + 2^{g_2(\sigma_0/16)\cdot n(1+o(1))} + \cdots \\
&= 2^{0.2059n(1+o(1))} + 2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots .
\end{aligned}
$$

$4\sigma_0 > 1$

cannot add $F(4\sigma_0)$

We also write the $2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots = o(2^{0.2059n(1+o(1))})$ terms explicitly.

$g_2(\sigma)$

Some numerical results for large p

There exists a threshold $p_0 \approx 2.1$ (we do not attempt to calculate the exact value of p_0) such that when $p > p_0$, $F_p(\sigma)$ attains its maximum at $\sigma = 1$. For $\sigma = 1$, i.e. $m = n$, we have another lower bound. Let $m = n$, and recall inequalities (3) and (5). We have

when p is large

$g_p(\sigma)$ is increasing

in $[0, 1]$.

$$
\begin{aligned}
A_p(n, 1/2) &\geq \sum_{i=1}^{r} |J_i'(n,n)| \\
&= |J_1'(n,n)| + \sum_{i=2}^{r} |J_i'(n,n)| \\
&\geq |J_1'(n,n)| + \sum_{i=2}^{r} \left\lceil \frac{\binom{n}{m_i} 2^{m_i}}{B_{i,n}(n)} \right\rceil \\
&= \boxed{|J_1'(n,n)|} + \sum_{i=2}^{r} F_p\left(\frac{1}{2^{p(i-1)}}\right).
\end{aligned}
$$

Indeed, we can improve the lower bound for $|J_1'(n,n)|$ slightly.

Relation between the ℓ_p distance and the Hamming distance

Recall the definition of $J_1(n,n)$ and $J_1'(n,n)$. $J_1(n,n) = \{\pm 1\}^n$ and $J_1'(n,n)$ is a largest subset of $\{\pm 1\}^n$ in which points have pairwise distance larger than or equal to $n^{1/p}$. For $\boldsymbol{u}, \boldsymbol{v} \in \{\pm 1\}^n$, let $d_H(\boldsymbol{u}, \boldsymbol{v}) := |\{i : u_i \neq v_i\}|$ be the Hamming distance between them. The following lemma is an easy observation.

Lemma 9

For every $\boldsymbol{u}, \boldsymbol{v} \in \{\pm 1\}^n$, we have

$$(d_p(\boldsymbol{u}, \boldsymbol{v}))^p = 2^p \cdot d_H(\boldsymbol{u}, \boldsymbol{v}).$$

☆ only valid for $J_1(n,n)$

$u \in J_2(n,n)$, the coordinates of u is $\frac{m}{m_2}, -\frac{m}{m_2}, 0,$

By this lemma, it suffices to find a largest subset of $\{\pm 1\}^n$, in which points have pairwise Hamming distance larger than or equal to $\lceil n/2^p \rceil$. Recall the definition of $B_{1,n}(\boldsymbol{u}, n)$ and we have

$$\begin{aligned} B_{1,n}(\boldsymbol{u}, n) &= \left\{ \boldsymbol{v} \in \{\pm 1\}^n : d_p(\boldsymbol{u}, \boldsymbol{v}) < n^{1/p} \right\} \\ &= \{ \boldsymbol{v} \in \{\pm 1\}^n : 2^p \cdot d_H(\boldsymbol{u}, \boldsymbol{v}) < n \} \\ &= \{ \boldsymbol{v} \in \{\pm 1\}^n : d_H(\boldsymbol{u}, \boldsymbol{v}) \leq \lceil n/2^p \rceil - 1 \} . \end{aligned}$$

So $B_{1,n}(n) = |B_{1,n}(\boldsymbol{u}, n)| = \sum_{k=0}^{\lceil n/2^p \rceil - 1} \binom{n}{k}$.

Improving the G-V bound by a factor of n

We have the following theorem, which gives a better lower bound for $|J_1'(n,n)|$ than that in inequality (5).

Theorem 10 (Jiang and Vardy, *IEEE Trans. Inform. Theory*, 2004)

There exists a positive constant c such that

$$|J_1'(n,n)| \geq c\frac{2^n}{B_{1,n}(n)}\log_2 B_{1,n}(n).$$

improvement of G-V bound.

$\sim n$

Note that

$$\lim_{n\to\infty}\frac{1}{n}\log_2 B_{1,n}(n) = H\left(\frac{1}{2^p}\right),$$

by Stirling's formula. So

$$|J_1'(n,n)| \geq c\frac{n2^n}{B_{1,n}(n)} = cn2^{n(1-H(2^{-p})+o(1))},$$

for some constant c (maybe depends on p).

For $p = 3$, we have

$$A_3(n, 1/2) \geq |J_1'(n,n)| + \sum_{i=2}^{r} F_3\left(\frac{1}{2^{3(i-1)}}\right)$$
$$\geq cn2^{n(1-H(2^{-3})+o(1))} + F_3(0.1250) + F_3(0.0156) + \cdots$$
$$\geq cn2^{n(1-H(2^{-3})+o(1))} + 2^{g_3(0.1250)\cdot n(1+o(1))} + 2^{g_3(0.0156)\cdot n(1+o(1))} + \cdots$$
$$= cn2^{0.4564n(1+o(1))} + 2^{0.1562n(1+o(1))} + 2^{0.0425n(1+o(1))} + \cdots.$$

For $p = 4$, we have

$$
\begin{aligned}
A_4(n, 1/2) &\geq |J_1'(n,n)| + \sum_{i=2}^{r} F_4\left(\frac{1}{2^{4(i-1)}}\right) \\
&\geq cn2^{n(1-H(2^{-4})+o(1))} + F_4(0.0625) + F_4(0.0039) + \cdots \\
&\geq cn2^{n(1-H(2^{-4})+o(1))} + 2^{g_4(0.0625)\cdot n(1+o(1))} + 2^{g_4(0.0039)\cdot n(1+o(1))} + \cdots \\
&= cn2^{0.6627n(1+o(1))} + 2^{0.1083n(1+o(1))} + 2^{0.0145n(1+o(1))} + \cdots .
\end{aligned}
$$

Further remarks

Sah et al. (*Adv. Math.*, 2020) obtained an inequality between ℓ_p-spherical codes for different p; that is, $A_p(n,d) \le A_q(n, d^{p/q})$ for all $1 \le q \le p$ and $d \in (0,1]$. So

$$A_2(n,d) \le A_p(n, d^{2/p}), \text{ if } 1 \le p < 2, \tag{7}$$

and

$$A_p(n,d) \le A_2(n, d^{p/2}), \text{ if } p \ge 2. \tag{8}$$

Sah et al. used inequality (8) to obtain an upper bound for $A_p(n,d)$ $(p \ge 2)$.

Lower bound for $A_2 \Rightarrow$ Lower bound for A_p,

upper bound for $A_2 \Rightarrow$ upper bound for A_p, On the other hand, we can use inequality (7) to obtain a lower bound for $A_p(n, 1/2)$ ($p \leq 2$) before. We need the following theorem, which is the best known lower bound for $A_2(n, d)$ ($d \in (0, 1)$).

Theorem 11 (Fernández et al., arXiv:2111.01255)

Let $\theta \in (0, \pi/2)$ be fixed. Then

$$A_2(n, \sin(\theta/2)) \geq (1 + o(1)) \ln \frac{\sin\theta}{\sqrt{2}\sin(\theta/2)} \cdot n \cdot \frac{\sqrt{2\pi n}\cos\theta}{\sin^{n-1}\theta}.$$

For $1 < p \leq 2$, we have

$$A_p(n, 1/2) \geq A_2(n, (1/2)^{p/2}).$$

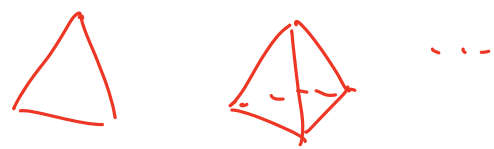
Let $\sin(\theta/2) = 2^{-p/2}$. Then $\cos(\theta/2) = \sqrt{1 - 2^{-p}}$, $\sin\theta = 2^{1-p/2}\sqrt{1 - 2^{-p}}$, and $\cos\theta = 1 - 2^{1-p}$. So

$$\begin{aligned} A_p(n, 1/2) &\geq A_2(n, (1/2)^{p/2}) \\ &= A_2(n, \sin(\theta/2)) \\ &\geq (1 + o(1)) \ln\sqrt{2 - 2^{1-p}} \cdot n \cdot \frac{\sqrt{2\pi n}(1 - 2^{1-p})}{(2^{1-p/2}\sqrt{1 - 2^{-p}})^{n-1}}. \end{aligned} \tag{9}$$

After some numerical calculations, when $p \in (1.9948, 2]$, the lower bound in inequality (9) is better than that in inequality (6).

Open problems

- Determine the kissing number of spheres for a specific dimension (e.g. $n = 5$). find lower and upper on Wiki, $40 \le \quad \le \cancel{44}$
- Shrink the gap between the lower and the upper bound for high dimensions. sphere: $[1.15\ldots^n, 1.3\ldots^n]$, l_p
- Find the bounds for kissing numbers of other convex bodies (e.g. the standard simplex). This is open for almost all convex bodies.



$\le 3^n - 1$

"=" iff parallelotope.

Thank you for your attention!