Log-Concavity of High Convolutions: The Odlyzko-Richmond Theorem

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MIT $\to$ Stanford

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Unimodality and Log-concavity

Definition

A sequence $a_1, \cdots, a_n$ is unimodal if there exists a K such that $a_1 \leq \cdots \leq a_K$ and $a_K \geq a_{K+1} \geq \cdots$.
A sequence $a_1, \cdots, a_n$ is log-concave if every term is non-negative and $a_k^2 \geq a_{k-1}a_{k+1}$ for all k.

Definition

A polynomial is unimodal / log-concave iff its coefficients are unimodal / log-concave.

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Definition

A polynomial is unimodal / log-concave iff its coefficients are unimodal / log-concave.

Example:

- $x^2 + 2$ is neither unimodal nor log-concave.
- $x^2 + x + 2$ is unimodal but not log-concave.
- $x^2 + 3x + 2$ is log-concave.

Catch Phrase

Unimodality and Log-concavity happen for reasons!

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Stanley, R.P.,
Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry.

Pak, I.,
What is a combinatorial interpretation?

Algebraic Reasons

Theorem

The product of two log-concave polynomials is a log-concave polynomial.

Corollary

Hyperbolic polynomials(polynomials with negative real roots) are log-concave.

Example: $(x+2)(x+3) = x^2 + 5x + 6$.

Example[Heilman-Lieb]: In graph G, let t_j be the number of matchings of size j. Then $\sum_j t_j x^j$ is hyperbolic, so $\{t_j\}$ is log-concave.

Geometric Reasons

Let K, L be convex bodies in $\mathbb{R}^n$. Let $f(t)$ be the volumn of $K + tL$.

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Let K, L be convex bodies in $\mathbb{R}^n$. Let $f(t)$ be the volumn of $K + tL$.

Theorem (Aleksandrov-Fenchel Inequality)

$f(t)$ is a log-concave polynomial.

A number of interesting corollaries in combinatorics.

Combinatorial Reasons

Theorem (Horrocks 2002)

Let $P_k(G)$ be the k-element **dependent** *sets in a graph G, and let $p_k(G) = \#P_k(G)$. Then $\{p_k(G)\}$ is log-concave.*

Proof by cleverly constructing injections.

Combinatorial Reasons

Theorem (Horrocks 2002)

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Proof by cleverly constructing injections.

Conjecture

Let $D_k(G)$ be the k-element **dominating** sets in a graph G, and let $d_k(G) = \#D_k(G)$. Then $\{d_k(G)\}$ is unimodal.

Algebraic Geometric Reasons

Theorem (Mason's conjecture)

The characteristic polynomial of a matroid is log-concave.

Key tools: Hodge theory, Lorentzian polynomials(June Huh et. al.), Completely Log-concave polynomials(Nima Anari et. al.)

Algebraic Geometric Reasons

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See Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant

Log-Concave Polynomials III: Mason's Ultra-Log-Concavity Conjecture for Independent Sets of Matroids.

Analytic Reasons(This talk)

Idea: Obtain a sufficiently accurate analytic estimate of a_k, for which $a_k^2 \geq a_{k-1}a_{k+1}$ follows.

Example(DeSalvo-Pak 2015)

Let $p(n)$ be the number of partitions of n.

$$1, 1, 2, 3, 5, 7, \cdots$$

Then $\{p(n)\}$ is log-concave for $n \geq 26$.

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Then $\{p(n)\}$ is log-concave for $n \geq 26$.

Theorem (Rademacher's Exact Formula)

$$p(n) = \frac{\sqrt{12}}{24n-1}\left(1 - \frac{1}{\pi\sqrt{24n-1}/6}\right)e^{\pi\sqrt{24n-1}/6} + R_n$$

where R_n is super-polynomially smaller than the first term.

Applications of Unimodality / Log-concavity

- Implications in theoretical computer science: unimodality can be exploited to design fast algorithms.
  Example: Proof of Mason's Conjecture gives FPRAS for counting the base of matroids.
- Connections with probability theory and statistics: log-concave distributions.
- Connections with Riemann Hypothesis: Pólya–Jensen criterion.

Methodology

Hardy-Littlewood Circle Method!

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The product of two log-concave polynomials is log-concave.
Thus, if $f(z)$ is log-concave polynomial, then $f(z)^N$ is log-concave for all $N \geq 1$.

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Can log-concavity **arise** simply by taking the power of polynomials?

Motivation

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Can log-concavity **arise** simply by taking the power of polynomials?

Trivial counterexample: $p(z) = z^2 + 1$. No power of p is log-concave.

Theorem (Odlyzko-Richmond, 1982)

If $p(z)$ is a polynomial with all positive coefficients, then $p^k(z)$ is log-concave for all sufficiently large k.

In fact, we only need all coefficients to be non-negative, and $p(z) \neq q(z^m)$ for any polynomial q and $m > 1$.

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Original title

On the Unimodality of High Convolutions of Discrete Distributions

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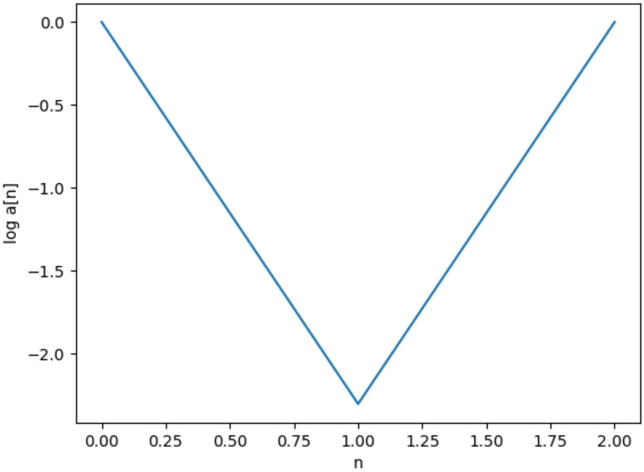
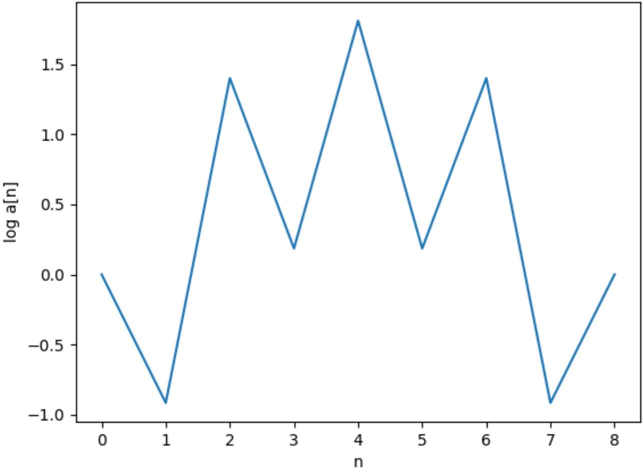
On the Unimodality of High Convolutions of Discrete Distributions

Example

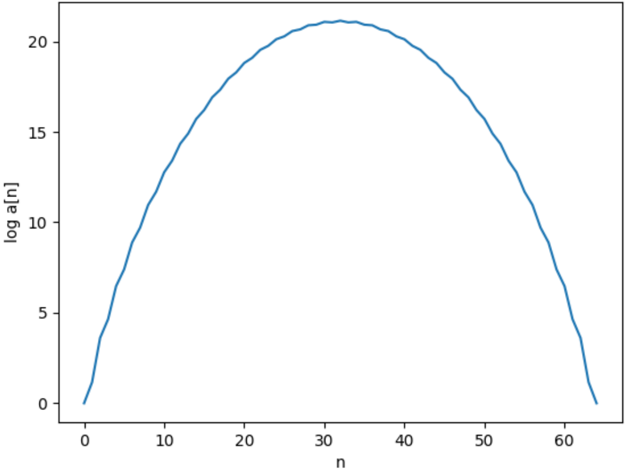
$p(z) = z^2 + 0.1z + 1$. Far from Log-concave.



Coefficients of $p(z)^4$



Coefficients of $p(z)^{32}$



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Notations

Theorem (Odlyzko-Richmond, 1982)

If $p(z)$ is a polynomial with all positive coefficients, then $p^k(z)$ is log-concave for all sufficiently large k.

Notation

Let $a_{k,n}$ be the coefficient of z^n in $p^k(z)$.

Assume: $n \leq dk/2$, and $n \geq k^{1/4}$.

d = degree of p(z).

Step 1: Set up Hardy-Littlewood Method

Let $a_{k,n}$ be the coefficient of z^n in $p^k(z)$.

Lemma

For any $r > 0$, we have

$$a_{k,n} = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} p^k(re^{i\theta}) e^{-in\theta} d\theta.$$

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More generally, let $I_r : \mathbb{R} \to \mathbb{R}$ be defined as

$$I_r(\alpha) = \int_{-\pi}^{\pi} p^k(re^{i\theta}) e^{-i\alpha\theta} d\theta$$

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Then

$$a_{k,n} = \frac{1}{2\pi r^n} I_r(n).$$

Step 1: Set up Hardy-Littlewood Method

To prove $a_{k,n}^2 \geq a_{k,n-1}a_{k,n+1}$, it suffices to find **one** $r > 0$ such that

$$2r^n a_{k,n} \geq r^{n-1}a_{k,n-1} + r^{n+1}a_{k,n+1}$$

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Strengthening: Find an $r > 0$ such that for any $\alpha \in [n-1, n+1]$,

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We can compute I_r'' explicitly.

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Step 1: Set up Hardy-Littlewood Method

Goal

Find an $r > 0$ such that for any $\alpha \in [n-1, n+1]$,

$$\int_{-\pi}^{\pi} \theta^2 p^k(re^{i\theta}) e^{-i\alpha\theta} d\theta \geq 0.$$

Technical Assumption: $r \sim n/k$, where the implied constant does not depend on n or k.

Step 2: Splitting into Major and Minor Arc

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Find an $r > 0$ such that for any $\alpha \in [n-1, n+1]$,

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Lemma

There exists a $c > 0$ independent of n, k such that

$$\left|p(re^{i\theta})\right| \leq p(r)e^{-c\theta^2 r}.$$

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Let $\theta_0 = k^{1/30}n^{-1/2}$. We call the region $|\theta| \leq \theta_0$ the **major arc**, and the region $|\theta| \in [\theta_0, \pi]$ the **minor arc**.

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Corollary (Estimate on Minor Arc)

For any $\epsilon > 0$, there exists a $c > 0$ such that for $r > \epsilon n/k$, we have

$$\left|\int_{|\theta|\in[\theta_0,\pi]} \theta^2 p^k(re^{i\theta})e^{-i\alpha\theta}d\theta\right| \leq p^k(r)e^{-ck^{1/15}}.$$

Step 3: Estimate on the Major Arc

Let $\theta_0 = k^{1/30}n^{-1/2}$. The integral over Major Arc is

$$\int_{|\theta|<\theta_0} \theta^2 p^k(re^{i\theta})e^{-i\alpha\theta}d\theta.$$

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Lemma

There exists constant $c > 0$ such that

$$\int_{|\theta|<\theta_0} \left|\theta^2 p^k(re^{i\theta})\right| d\theta \geq c p^k(r) \cdot k^{-4}.$$

Goal: For $|\theta| < \theta_0$, ensure that

$$\left|\arg p^k(re^{i\theta}) e^{-i\alpha\theta}\right| < \frac{\pi}{4}$$

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$$\arg p^k(re^{i\theta})e^{-i\alpha\theta} = k \arg p(re^{i\theta}) - \alpha\theta.$$

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For $|\theta| < \theta_0$, ensure that

$$\left|\arg p^k(re^{i\theta})e^{-i\alpha\theta}\right| < \frac{\pi}{4}.$$

$$\arg p^k(re^{i\theta})e^{-i\alpha\theta} = k \arg p(re^{i\theta}) - \alpha\theta.$$

Since $\arg p(re^{i\theta})$ is an **odd function** in θ, its Taylor expansion is

$$\arg p(re^{i\theta}) = \frac{rp'(r)}{p(r)} \cdot \theta + O\left(r\theta^3\right)$$

Recall that $|\theta| < k^{1/30}n^{-1/2}$, so take r such that $rp'(r)/p(r) = n/k$. We can verify that such an r exists and $r \sim n/k$.

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Counterexample for Power Series

Theorem (Odlyzko-Richmond, 1982)

If $p(z)$ is a polynomial with all positive coefficients, then $p^k(z)$ is log-concave for all sufficiently large k.

Does this hold for power series?

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Does this hold for power series?

No. Example: $p(z) = \sum_{n=0}^{\infty} z^{2^n} + \sum_{n=0}^{\infty} \epsilon_n z^n$, for fast decaying $\epsilon_n > 0$.

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Question

Can we prove sufficient condition on power series $p(z)$ for analogs of Odlyzko-Richmond to hold?

Motivation in Recent Research

Definition (Nekrasov-Okounkov Polynomials)

Let $Q_n(z)$ be the polynomials satisfying

$$\sum_{n=0}^{\infty} Q_n(z)q^n = \prod_{n=0}^{\infty} (1 - q^n)^{-z-1}.$$

An important family of polynomials in algebraic combinatorics, with connections to Young Tableaux.

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An important family of polynomials in algebraic combinatorics, with connections to Young Tableaux.

$$1, 1+z, 2+\frac{5}{2}z+\frac{1}{2}z^2, 3+\frac{29}{6}z+2z^2+\frac{1}{6}z^3, \cdots$$

Heim-Neuhauser's Conjecture

Conjecture(Heim-Neuhauser 2018)

$Q_n(z)$ is unimodal for all n.

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A lot of inequalities later...

Lemma (Hong-Z. 2020)

Let $p(z) = \sum_{n\geq 1} \sigma_{-1}(n) z^n$, let $a_{k,n}$ be the coefficient of z^n in $p^k(z)$.

If there exists $C > 1$ such that $\{a_{k,n}\}_{n=1}^{\infty}$ is log-concave for $n \leq C^k$, then $Q_n(z)$ is unimodal for all sufficiently large n.

Reduction to Odlyzko-Richmond type result

Conjecture(Hong-Z. 2020)

Let $p(z) = \sum_{n \geq 1} \sigma_{-1}(n) z^n$, let $a_{k,n}$ be the coefficient of z^n in $p^k(z)$.

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There exists $C > 1$ such that $\{a_{k,n}\}_{n=1}^{\infty}$ is log-concave for $n \leq C^k$.

Numerical evidence: first n such that $a_{k,n}^2 < a_{k,n-1} a_{k,n+1}$.

k	2	3	4	5	6	7	8	9	10	11	12	13
$n_0(k)$	6	21	39	73	135	251	475	917	1801	3595	7259	14787

Resolution of Conjecture

Theorem (Z. 2022)

Let $p(z) = \sum_n a_n z^k$ *be a power series with*
1) $a_n \geq 1$ *for every* n*.*
2) there exists $A > 0, \alpha \in (0,1)$ *such that*

$$0 \leq A(n+1) - (a_0 + \cdots + a_n) \leq O((n+1)^\alpha).$$

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Richmond-Odlyzko method works for such $p(z)$. Gives $n \leq C^{k^{1/3}}$.
Additional considerations give $n \leq C^k$.

An Open Problem

Does there exist a combinatorial proof for Odlyzko-Richmond?

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