On the Maximum F_5-free Subhypergraphs of $G^3(n, p)$

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Joint work with Igor Araujo and József Balogh

Recently there has been a trend in Combinatorics to prove that certain known theorems are still valid in the *random sparse* setting.

- Sparse counting lemma
- Szemerédi's theorem $\rightarrow$ Green-Tao theorem

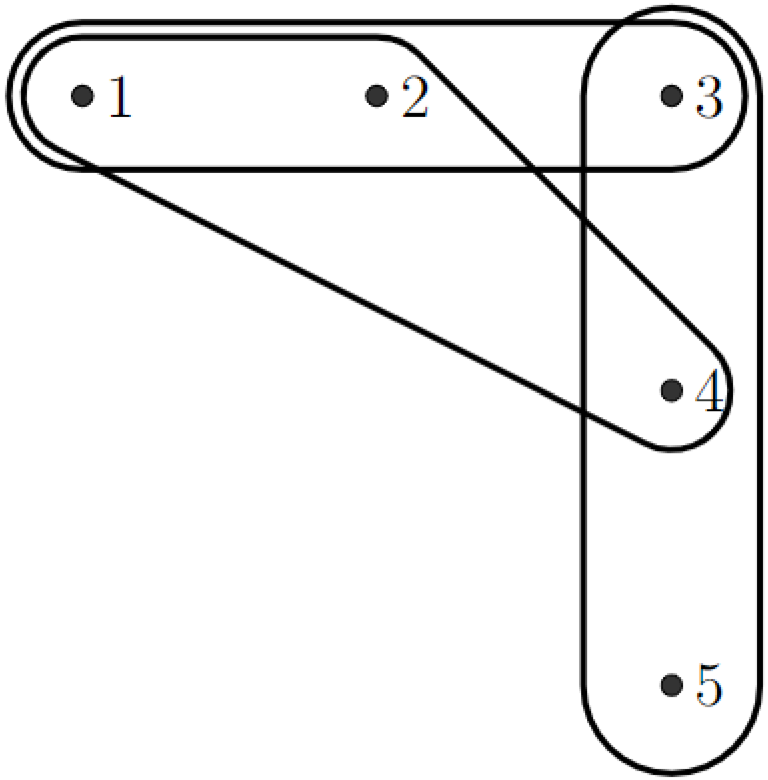
Theorem (Mantel)

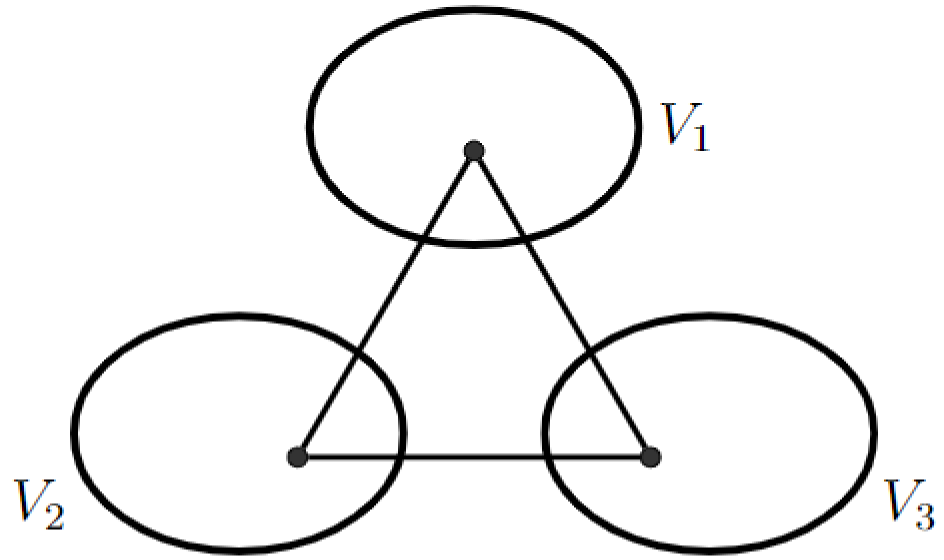
Every maximum triangle-free subgraph of K_n is bipartite.

Theorem (DeMarco, Kahn (2014))

There is a C such that if $p > Cn^{-1/2} \ln^{1/2} n$, with high probability every maximum triangle-free subgraph of $G(n, p)$ is bipartite.

The hypergraph F_5.





Theorem (Frankl and Füredi (1983), Keevash and Mubayi (2004))

For large enough n, every maximum F_5-free subhypergraph of K_n^3 is tripartite.

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For large enough n, every maximum F_5-free subhypergraph of $K_n^{(3)}$ is tripartite.

⬇

Theorem (Balogh, Butterfield, Hu, and Lenz (2015))

There is a K such that if $p > K \ln n/n$, with high probability every maximum F_5-free subhypergraph of $G^3(n, p)$ is tripartite.

⬇

Not a threshold!

Theorem (Araujo, Balogh, and L. (2022))

There is a K such that if $p > K\sqrt{\ln n}/n$, with high probability every maximum F_5-free subhypergraph of $G^3(n, p)$ is tripartite.

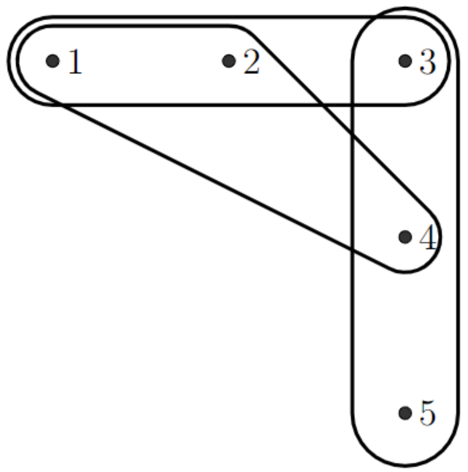
First, let look at the deterministic case($p = 1$).

- Stability theorem + cleaning

Theorem (Keevash and Mubayi (2004))

For any $\varepsilon > 0$, there exists $\delta > 0$ such that if H is an n-vertex F_5-free hypergraph with at least $(1-\delta)\frac{n^3}{27}$ hyperedges, then there is a partition of the vertex set of H as $V(H) = V_1 \cup V_2 \cup V_3$ such that all but at most εn^3 hyperedges of H have exactly one vertex in each V_i.

Cleaning



What if $p < 1$?

- Concentration
- Stability theorem + cleaning

Concentration

Lemma (Chernoff bound)

Let Y be the sum of mutually independent indicator random variables, and let $\mu = \mathbb{E}[Y]$. For every $\varepsilon > 0$, we have

$$\mathbb{P}[|Y - \mu| > \varepsilon\mu] < 2e^{-c_\varepsilon \mu},$$

where $c_\varepsilon = \min\left\{-\ln\left(e^\varepsilon(1+\varepsilon)^{-(1+\varepsilon)}\right), \varepsilon^2/2\right\}$.

For example, the degree of every vertex is about $n^2p/2$.

Stability

Theorem (Keevash and Mubayi (2004))

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Theorem (Samotij (2014))

For any $\varepsilon > 0$, there exists $\delta > 0$ and $C > 0$ such that if $p > C/n$ and H is an n-vertex F_5-free subhypergraph of $G^3(n,p)$ with at least $(1-\delta)\frac{n^3}{27}p$ hyperedges, then there is a partition of the vertex set of H as $V(H) = V_1 \cup V_2 \cup V_3$ such that all but at most $\varepsilon n^3 p$ hyperedges of H have exactly one vertex in each V_i.

Cleaning

Everything seems to be good... until p is $K/\sqrt{n}$.

Let H be a maximum F_5-free subhypergraph of $G^3(n,p)$. Let partition π be the 3-partition maximizing $|H_\pi|$.

$$p > \frac{K}{\sqrt{n}} \quad \to \quad p > \frac{K \ln n}{n}$$

- 3^n is not necessary. We do NOT need to know exactly which part every vertex belongs to.
- We only need to know about the number of hyperedges between them.

$$p > \frac{K \ln n}{n} \quad \to \quad p > \frac{K\sqrt{\ln n}}{n}$$

- The codegree of pairs of vertices.

Lemma

There exists a constant K such that if $p > K \ln n/n$, then with high probability the codegree of any pair of vertices in $G^3(n,p)$ is at most $3pn$.

⬇

Lemma

There exists a constant K such that if $p > K\sqrt{\ln n}/n$, then with high probability

- *the codegree of any pair of vertices in $G^3(n,p)$ is at most $pn\sqrt{\ln n}/\ln \ln n$, and*
- *the number of pairs of vertices with codegree more than $3pn$ in $G^3(n,p)$ is at most $n^2 e^{-\sqrt{\ln n}}$.*

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$$\mathbb{P}[|Y-\mu| > \varepsilon\mu] < 2e^{-c_\varepsilon \mu},$$

Conlon, D.; **Gowers, W. T.** Combinatorial theorems in sparse random sets. *Ann. of Math. (2)* **184** (2016), no. 2, 367--454. MR3548529

Schacht, Mathias. Extremal results for random discrete structures. *Ann. of Math. (2)* **184** (2016), no. 2, 333--365. MR3548528

Theorem (Samotij (2014))

For any $\varepsilon > 0$, there exists $\delta > 0$ and $C > 0$ such that if $p > C/n$ and H is an n-vertex F_5-free subhypergraph of $G^3(n, p)$ with at least $(1 - \delta)\frac{n^3}{27}p$ hyperedges, then there is a partition of the vertex set of H as $V(H) = V_1 \cup V_2 \cup V_3$ such that all but at most $\varepsilon n^3 p$ hyperedges of H have exactly one vertex in each V_i.

Samotij, Wojciech. Stability results for random discrete structures.
Random Structures Algorithms **44** (2014), no. 3, 269--289.

$$m_\ell(H) = \max\left\{\frac{e(K) - 1}{v(K) - \ell} : K \subseteq H \text{ with } v(K) \geq \ell + 1\right\}.$$

Thank you!