Turán number of the linear 3-graph Crown

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Overview

1. Introduction and definition
2. Our main result
3. Remaining Problems

Definition

Definition 1 Linear 3-graph

A **linear** 3-**graph**, also known as **linear triple system**, $H = (V, E)$ consists of a vertex set $V = V(G)$ and an edge set $E = E(G)$ of 3-element subsets of V, such that any two edges in E share at most one vertex.

Definition 2 Linear Turán number

For a linear 3-graph F, and a positive integer n, the **linear Turán number** $\mathrm{ex}(n, F)$ is the maximum number of edges in any F-free linear 3-graph on n vertices.

Introduction

Theorem 1 (6,3)-theorem (Ruzsa and Szemerédi, 1976)

Let T be the linear 3-graph triangle, $\frac{n^2}{e^{O(\sqrt{\log n})}} \leq ex(n, T) \leq o(n^2)$.

The (6,3)-theorem has a huge influence, for example, the celebrated triangle removal theorem is devised in order to find another proof of it. A recent direction is the linear Turán number of small trees. For example, the Turán number of B_4 and P_4 are solved by Gyárfás, Ruszinkó and Sárközy.

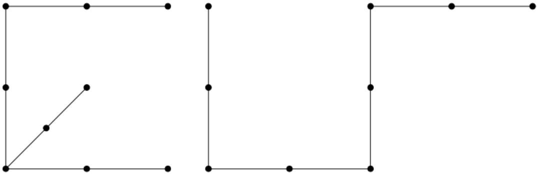

Figure: B_4 and P_4

Introduction

Theorem 2 (Gyárfás, Ruszinkó and Sárközy, 2021)

Let C_{13}, also called crown, be the linear 3-graph on 9 vertices $\{a, b, c, d, e, f, g, h, i\}$ with edges $E = \{\{a, b, c\}, \{a, d, e\}, \{b, f, g\}, \{c, h, i\}\}$. Then $6\lfloor \frac{n-3}{4} \rfloor \leq \mathrm{ex}(n, C_{13}) \leq 2n$.

Fletcher imporve the upper bound.

Theorem 3 (Fletcher, 2021)

$\mathrm{ex}(n, C_{13}) \leq \frac{5}{3}n$.

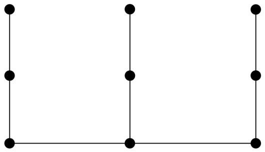

Figure: Crown

Main Theorem

Theorem 4 (Our first main theorem)

Let G be any crown-free linear 3-graph G on n vertices. Then its number of edges satisfies

$$|E(G)| \leq \frac{3(n-s)}{2}.$$

where s is the number of vertices in G with degree at least 6.

Theorem 5 (Our second main theorem)

Let G be any crown-free linear 3-graph G on n vertices, and let s be the number of vertices in G with degree at least 6. If $s \leq 2$, then the number of edges satisfies

$$|E(G)| \leq \frac{10(n-s)}{7}.$$

Main Theorem

Corollary 1

If $n \geq 63$, then

$$\mathrm{ex}(n, C_{13}) \leq \frac{3(n-3)}{2}.$$

Proof.

□

Proof of Theorem 4

Definition 3

Let G be a linear 3-graph, $\forall e = \{x, y, z\} \in E(G)$, let $D(e)$ denote the degree vector $\langle d(x), d(y), d(z) \rangle$ of e with $d(x) \geq d(y) \geq d(z)$. Furthermore, define a partial order on these vector $D(e) \geq D(f)$ if all coordinates of e is larger than or equal to f.

Definition 4

We call vertex v a large (degree) vertex if $d(v) \geq 6$, otherwise we call it a small (degree) vertex.

Proof of Theorem 4

Suppose G is the minimal crowm-free linear 3-graph such that G has greater than $3(n-s)/2$ edges. We use discharge method to find contradiction.

Give every small degree vertex charge 1, and uniformly distribute the charge on v to edges incident with it. So the total edge charge is equal to the vertex's, that is

$$
\begin{aligned}
\chi(v) =& 1, \\
\chi(e) =& \sum_{v\in e} \frac{\chi(v)}{d(v)}. \\
\sum_{e\in E(G)} \chi(e) =& \sum_{e\in E(G)} \sum_{v\in e} \frac{\chi(v)}{d(v)} = \sum_{v\in V(G)} \sum_{e\ni v} \frac{\chi(v)}{d(v)} \\
=& \sum_{v\in V(G)} \chi(v) = n - s.
\end{aligned}
$$

Proof of Theorem 4

Since $\frac{2}{3}|E(G)| > n - s$, by Pigeonhole Principle, there exists one edge $e_0 = \{x_0, y_0, z_0\}$ such that

$$\chi(e_0) = \frac{\chi(x_0)}{d(x_0)} + \frac{\chi(y_0)}{d(y_0)} + \frac{\chi(z_0)}{d(z_0)} < \frac{2}{3}. \tag{1}$$

WLOG, let $D(e_0) = \langle d(z_0), d(y_0), d(x_0) \rangle$.

Claim:

$$D(e_0) \geq \langle 5, 5, 4 \rangle. \tag{2}$$

Proof of Theorem 4

proof of claim:

Proof of Theorem 4

Lemma 1

Let G be a crown-free graph and $e = \{x, y, z\}$ satisfy $D(e) \geq \langle 5, 5, 4 \rangle$. Then, the vertex set of all vertices sharing an edge with $\{x, y, z\}$,

$$S = \bigcup_{f \in E(G), f \cap \{x,y,z\} \neq \emptyset} f,$$

contains exactly 11 vertices and all vertices in S have degree at most 5. The set of edges that contains at least one vertex in S,

$$E_S = \{f : f \in E(G), f \cap S \neq \emptyset\},$$

contains at most 13 edges, and all elements of E_S are subsets of S. In other words, the subgraph $G[S]$ is a connected component of G.

Proof of Theorem 4

Let $G - S$ be the graph obtained by deleting the vertices S and the edges in E_S.
By the lemma, the graph $G - S$ has $n' = n - 11$ vertices and at least $|E(G)| - 13$ edges. Furthermore, the number of vertices in $G - S$ of degree at least 6 is exactly s.
Therefore, we conclude that

$$|E(G - S)| \geq |E(G)| - 13 > \frac{3(n - s)}{2} - 13 > \frac{3(n' - s)}{2},$$

contradicting the assumption that G is the smallest counterexample.
So we have shown **Theorem 4**.□

Proof of Theorem 5

Remark: The proof of **Theorem 5** is similar to **Theorem 4**, the only difference is that we use new discharging method to the vertices with degree 3 since there are at most 2 large vetices.

Proof of Theorem 5

Suppose G is the minimal crowm-free linear 3-graph such that G has greater than $10(n-s)/7$ edges. Also use discharge method to find contradiction.

Give every small degree vertex charge 1, and let

$$\chi(v,e)=\begin{cases}1, & \textit{if } d(v)<6 \textit{ and } d(v)\neq 3,\\ 1.05, & \textit{if } d(v)=3 \textit{ and } \exists u\in e \textit{ s.t. } d(u)\geq 6,\\ 0.9, & \textit{if } d(v)=3 \textit{ and } \nexists u\in e \textit{ s.t. } d(u)\geq 6.\end{cases}$$

$$\chi(e)=\sum_{v\in e}\frac{\chi(v,e)}{d(v)}.$$

Similarly, there exists one edge $e_0=\{x_0,y_0,z_0\}$ such that

$$\chi(e_0)=\frac{\chi(x_0)}{d(x_0)}+\frac{\chi(y_0)}{d(y_0)}+\frac{\chi(z_0)}{d(z_0)}<\frac{7}{10}.$$

Proof of Theorem 5

Claim:

$$D(e_0) \geq \langle 5, 5, 4 \rangle.$$

proof of claim:

Proof of Theorem 5

Same as **Theorem 4**, using **Lemma 1** to obtain

$$|E(G-S)| \geq |E(G)| - 13 > \frac{3(n-s)}{2} - 13 > \frac{3(n'-s)}{2},$$

which is contradiction.$\square$

First we observe the elements of S. For any $p \in \{x, y, z\}$, Define

$$G(p) = \{q : q \in V(G), q\ p\ \} \backslash \{x, y, z\},$$
$$E(p) = \{f : f \in E(G), f\ p \quad f \neq e\},$$

We can observe that $d(z) = d(y) = 5$, so$|G(z)| = |G(y)| = 8, |G(x)| \geq 6$, $|E(z)| = |E(y)| = 4, |E(x)| \geq 3$.

Claim:

$$G(y) \subset G(z). \tag{3}$$

Proof of Lemma 1

proof of claim:

Proof of Lemma 1

Similarly, $G(z) \subset G(y), G(x) \subset G(z)$. So $S \backslash \{x, y, z\} = G(z) = G(y) \supset G(x)$. Furtherly define F as the set of all edges in $E(G)$ that contains one of the vertices in S, but is disjoint from $\{x, y, z\}$, but is disjoint from $\{x, y, z\}$, that is

$$F = \{f : f \in E(G), f \cap G(z) \neq \emptyset \; f \cap \{x, y, z\} = \emptyset\}.$$

Now the remaining proof suffices to show that F must be empty.
We denote the vertices in $G(z)$ by a, b, c, d, r, s, p, q, such that $\{z, a, b\}, \{z, c, d\}, \{z, r, s\}, \{z, p, q\}$ are edges in $E(G)$. Now we follow three steps to prove the statement.

Proof of Lemma 1

Step 1, we construct a auxillary bipartitie graph $H = (X_H, Y_H, E_H)$, where

$$X_H = \{e_i | y \in e_i\}, Y_H = \{e_j | z \in e_j\}, E_H = \{\{e_i, e_j\} | e_i \cap e_j \neq \emptyset\}$$

.

H is a 2-regular bipartite graph with order 8. Thus, $H = C_8$ or $C_4 \biguplus C_4$.

Claim:

$$H \text{ contains a } K_{2,2}. \tag{4}$$

Proof of Lemma 1

proof of claim:

Proof of Lemma 1

Thus $H = C_4 \uplus C_4$. Step 2, We claim that there exists no edge containing x that contains exactly one vertex in V_1 and another one in V_2.

proof of claim:

Proof of Lemma 1

Step 3, let f be any element of F. We do the discussion about elements in f. By symmetry we can let $a \in f$. Then we can see $b, c \notin f$.

Firstly, we claim that f cannot contain exactly one element a of S.

Secondly, we claim that $d \notin f$.

Therefore, we can assume $r \in f$ by symmetry.

proof

Remaining Problems

We introduce the (6,3)-theorem in the first place. In fact, the (6,3)-family only have one elements, while the (7-4)-family have three elements. A more generally conjecture is shown below.

Conjecture 1

If a linear triple system on n vertices does not contain any member of (k+3, k)-family then it has $o(n^2)$ triples.

The special case of conjecture 1 when $k = 4$ is also a good question remain to be solved.

For the time being, the following theorem is the best result.

Theorem 6(Gyárfás, Sárközy, 2020)

If a linear triple system on n vertices does not contain any member of (k+2+$\lfloor \log_2 k \rfloor$, k)-family then it has $o(n^2)$ triples.

Remaining Problems

On the other hand, although our main theorem has completed the determination of linear Turán number for 3-trees with at most 4 edges, there are also other Turán number for 3-trees remain to be determined.

- Question : What is the linear Turán number of k-edge linear path P_k?

For the time being, we have the following theorem, but it is said to be 'far from best possible' by the auther.

Theorem 7(Gyárfás, Ruszinkó, Sárközy, 2021)

$\mathrm{ex}(n, P_k) \leq 1.5kn$.

In the case of P_4, $\mathrm{ex}(n, P_4) \leq \frac{4}{3}n$, with equality only for the disjoin union of affine plane of order 3.

The End