Relaxations of coloring squares of graphs

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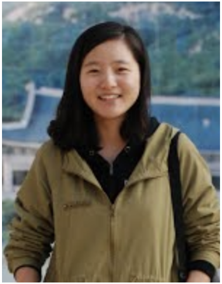
Hankuk University of Foreign Studies (HUFS)

Joint work with


Eun-Kyung Cho


Boram Park


Hyemin Kwon

proper coloring: each edge is rainbow

The chromatic number $\chi(G)$ is the min k where G is k-colorable.

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$$\chi(G) \leq \Delta(G) + 1$$

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proper conflict-free coloring: each neighborhood has a unique color

odd coloring: each neighborhood has an odd color

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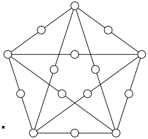
$$\chi(K_c^*) = 2 \leq c = \chi_{\mathrm{odd}}(K_c^*) = \chi_{\mathrm{pcf}}(K_c^*)$$



K_c^*: graph obtained by subdividing every edge of K_c exactly once.

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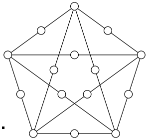

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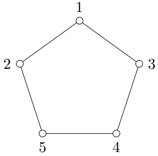
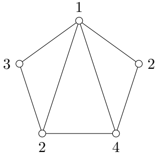
not monotone:





12.27 Petruševski, Škrekovski: "Colorings with neighborhood parity condition"

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01.05 Cranston: "Odd Colorings of Sparse Graphs"
01.10 Caro, Petruševski, Škrekovski: "Remarks on odd colorings of graphs"
01.28 Petr, Portier: "The odd chromatic number of a planar graph is at most 8"
02.05 (Cranston,) Lafferty, Song: "A Note on Odd Colorings of 1-Planar Graphs"
02.05 Fabrici, Lužar, Rindošová, Soták:
"Proper conflict-free and unique-maximum colorings of planar graphs w.r.t neighborhoods"
02.23 Cho, C., Kwon, Park: "Odd coloring of sparse graphs and planar graphs"
02.25 Dujmović, Morin, Odak: "Odd Colourings of Graph Products"
03.02 Caro, Petruševski, Škrekovski: "Remarks on proper conflict-free colorings of graphs"
03.19 Hickingbotham: "Odd colouring, conflict-free colouring and strong colouring number"
03.23 Liu: "Proper conflict-free list-coloring, subdivisions, and layered treewidth"
03.30 Cho, C., Kwon, Park: "Proper conflict-free coloring of sparse graphs"
05.09 Metrebian: "Odd colouring on the torus"
05.19 Qi, Zhang: "Odd coloring of two subclasses of planar graphs"
06.12 Tian, Yin: "The odd chromatic number of a toroidal graph is at most 9"
06.13 Tian, Yin: "Every toroidal graph without 3-cycles is odd 7-colorable"
06.15 Tian, Yin: "Every toroidal graphs without adjacent triangles is odd 8-colorable"
06.28 Liu, Wang, Yu: "1-planar graphs are odd 13-colorable"
10.06+ Cho, C., Kwon, Park: "Odd/proper conflict-free coloring graphs w bounded Δ"

Conjecture (2021.12+ Petruševski, Škrekovski)

If G is planar, then $\chi_{\mathrm{odd}}(G) \leq 5$. *Tight for 5-cycle.*

Theorem (2021.12+ Petruševski, Škrekovski)

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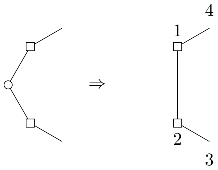

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partitionable into four odd forests

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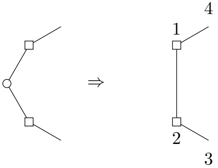

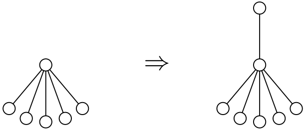

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Conjecture (2022.01+ Caro, Petruševski, Škrekovski)

If G has $\Delta(G) \geq 3$, then $\chi_{\mathrm{odd}}(G) \leq \Delta(G) + 1$.

Conjecture (2021.12+ Petruševski, Škrekovski)

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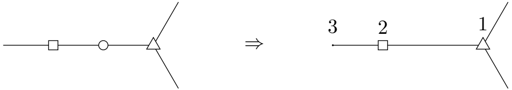
True for $\Delta(G) = 3$.



maximum average degree: $\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$

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$c \in \{1, 2\}$: if $\mathrm{mad}(G) \leq c - 1$, then $\chi_{\mathrm{odd}}(G) \leq c$. Tight for P_2, P_3.

$c \in \{3, 4\}$: if $\mathrm{mad}(G) < 2$, then $\chi_{\mathrm{odd}}(G) \leq 3 \leq c$. Tight for C_5.

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Conjecture (2022.01+ Cranston)

$c \geq 4$: if G has $\text{mad}(G) < \text{mad}(K^*_{c+1})$, then $\chi_{\text{odd}}(G) \leq c$.

Theorem (2022.01+ Cranston)

$c \geq 4$: if G has $\text{mad}(G) < \text{mad}(K^*_{c+1})$, then $\chi_{\text{odd}}(G) \leq c+3$.

$c \in \{5,6\}$: if G has $\text{mad}(G) \leq \text{mad}(K^*_{c+1})$, then $\chi_{\text{odd}}(G) \leq c$, unless $K^*_{c+1} \subseteq G$.

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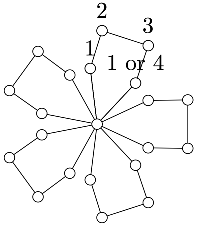

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$c = 4$: Conj. is FALSE!

$\mathrm{mad}(H_k) = \frac{10k}{4k+1} \le \frac{8}{3} = \mathrm{mad}(K^*_4)$ for all k.

$\chi_{\mathrm{odd}}(H_k) > 4$. $C_5 \subseteq H_k$ and C_5 is H_1.

maximum average degree: $\mathrm{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$

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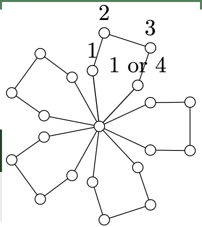

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$c = 4$: Conj. is FALSE! $\mathrm{mad}(H_k) = \frac{10k}{4k+1} \leq \frac{8}{3} = \mathrm{mad}(K^*_4)$ for all k. $\chi_{\mathrm{odd}}(H_k) > 4$. $C_5 \subseteq H_k$ and C_5 is H_1.

Theorem (2022.02+ Cho, C., Kwon, Park)

$c \geq 7$: *if G has $\mathrm{mad}(G) \leq \mathrm{mad}(K^*_{c+1})$, then $\chi_{\mathrm{odd}}(G) \leq c$, unless $K^*_{c+1} \subseteq G$.*

$c = 4$: *if G has $\mathrm{mad}(G) < \frac{22}{9}$ and no induced 5-cycle, then $\chi_{\mathrm{odd}}(G) \leq 4$.*

Natural corollaries to planar graphs with girth restrictions.

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Corollary (2022.01+ Cranston, 2022.02+ Cho, C., Kwon, Park)

$c \geq 5$: *if G is planar with girth* $\geq \frac{4c}{c-2}$, *then* $\chi_{\mathrm{odd}}(G) \leq c$.
$c = 4$: *if G is planar with girth* ≥ 11, *then* $\chi_{\mathrm{odd}}(G) \leq 4$.

Corollary implies

planar with girth ≥ 5 has $\chi_{\mathrm{odd}}(G) \leq 10$
planar with girth ≥ 6 has $\chi_{\mathrm{odd}}(G) \leq 6$
planar with girth ≥ 7 has $\chi_{\mathrm{odd}}(G) \leq 5$

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Theorem (2022.02+ Cho, C., Kwon, Park)

If G is planar with girth ≥ 5, then $\chi_{\mathrm{odd}}(G) \leq 6$.

Natural corollaries to planar graphs with girth restrictions.

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Theorem (2022.02+ Cho, C., Kwon, Park)

If G is planar with girth ≥ 5, *then* $\chi_{\mathrm{odd}}(G) \leq 6$.

Theorem (2022.06+ Tian, Yin)

If G is planar with girth ≥ 4, *then* $\chi_{\mathrm{odd}}(G) \leq 7$.

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If G is planar with girth ≥ 5, then $\chi_{\mathrm{odd}}(G) \leq 6$.

Theorem (2022.06+ Tian, Yin)

If G is planar with girth ≥ 4, then $\chi_{\mathrm{odd}}(G) \leq 7$.

Remark:
"If G is planar with girth ≥ 6, then $\chi_{\mathrm{odd}}(G) \leq 4$."
implies Four Color Theorem!

Natural corollaries to planar graphs with girth restrictions.

Corollary (2022.01+ Cranston, 2022.02+ Cho, C., Kwon, Park)

$c \geq 5$: if G is planar with girth $\geq \frac{4c}{c-2}$, then $\chi_{\mathrm{odd}}(G) \leq c$.
$c = 4$: if G is planar with girth ≥ 11, then $\chi_{\mathrm{odd}}(G) \leq 4$.

Corollary implies

planar with girth ≥ 5 has $\chi_{\mathrm{odd}}(G) \leq 10$
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planar with girth ≥ 7 has $\chi_{\mathrm{odd}}(G) \leq 5$

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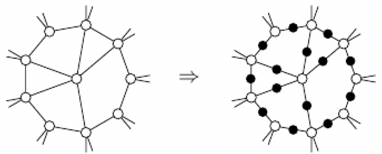
Theore

If G is ... $G) \leq 7$.



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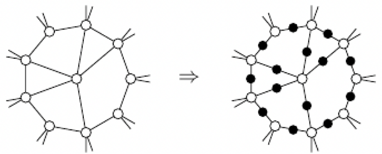

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What about proper conflict-free coloring?

Theorem (2022.02+ Fabrici Lužar, Rindošová, Soták)

If G is planar, then $\chi_{\mathrm{pcf}}(G) \leq 8$.

Conjecture (2022.02+ Fabrici Lužar, Rindošová, Soták)

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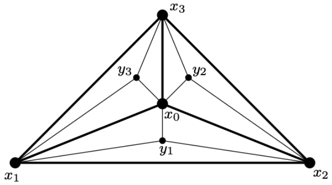

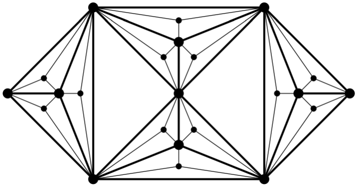

If there is a proper conflict-free coloring with 5 colors.....

– x_0, x_1, x_2, x_3 are colored 1,2,3,4

– at least one of y_1, y_2, y_3 are colored 5

– two of x_1, x_2, x_3 sees all colors!

$c \in \{1, 2\}$: if $\mathrm{mad}(G) \leq c - 1$, then $\chi_{\mathrm{pcf}}(G) \leq c$. Tight for P_2, P_3.

$c \in \{3, 4\}$: if $\mathrm{mad}(G) < 2$, then $\chi_{\mathrm{pcf}}(G) \leq 3 \leq c$. Tight for C_5.

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Theorem (2022.03+ Caro, Petruševski, Škrekovski)

If G has $\mathrm{mad}(G) < \frac{8}{3}$, then $\chi_{\mathrm{pcf}}(G) \le 6$.
If G has $\mathrm{mad}(G) < \frac{5}{2}$, then $\chi_{\mathrm{pcf}}(G) \le 5$.
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Corollary implies

planar with girth ≥ 8 has $\chi_{\text{pcf}}(G) \le 6$
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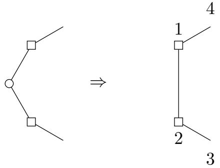

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Theorem (2022.03+ Cho, C. Kwon, Park)

$c \ge 5$: *if G has $\mathrm{mad}(G) \le \mathrm{mad}(K^*_{c+1})$, then $\chi_{\mathrm{pcf}}(G) \le c$, unless $K^*_{c+1} \subseteq G$.*
$c = 4$: *if G has $\mathrm{mad}(G) < \frac{12}{5}$ and no induced C_5, then $\chi_{\mathrm{pcf}}(G) \le 4$.*

Theorem implies planar with girth ≥ 5 has $\chi_{\mathrm{pcf}}(G) \le 10$

Theorem (2022.03+ Cho, C. Kwon, Park)

If G is planar with girth ≥ 5, then $\chi_{\mathrm{pcf}}(G) \le 7$.

Brooks-type.

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Conjecture (2022.01+ 2022.03+ Caro, Petruševski, Škrekovski)

If G has $\Delta(G) \geq 3$, then $\chi_{\mathrm{odd}}(G) \leq \Delta(G) + 1$ and $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 1$.

Theorem (2022.01+, 2022.03+ Caro, Petruševski, Škrekovski)

If G is connected, then $\chi_{\mathrm{odd}}(G) \leq 2\Delta(G)$, unless G is C_5
If G is connected, then $\chi_{\mathrm{pcf}}(G) \leq \lfloor 2.5\Delta(G) \rfloor$, equality iff $G \in \{K_2, C_5\}$.
If G is claw-free or chordal, then $\chi_{\mathrm{pcf}}(G) \leq 2\Delta(G) + 1$.

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If G has $lcc(G) \leq \ell$, *then* $\chi_{\mathrm{odd}}(G) \leq \frac{2\ell-1}{\ell}\Delta(G) + 2$.

If G is claw-free, then $\chi_{\mathrm{odd}}(G) = \chi_{\mathrm{pcf}}(G) \leq \frac{3}{2}\Delta(G) + \sqrt{\Delta(G)} + 1$.

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If G has $\Delta(G) \geq h + 2 \geq 3$, then $\chi^h_{\mathrm{pcf}}(G) \leq (h+1)\Delta(G) - 1$.
For infinitely many h, there is G such that $\chi^h_{\mathrm{pcf}}(G) = (h+1)(\Delta(G) - 1)$.

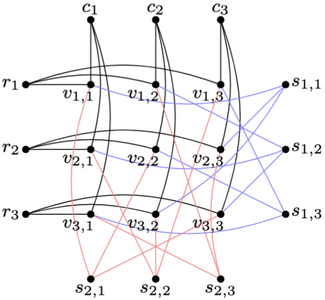
h-proper conflict-free coloring: each neighborhood has h "unique" colors

1	2	3
2	3	1
3	1	2

1	3	2
2	1	3
3	2	1

Orthogonal Latin squares of order n

n is a prime power
$\Delta = n + 1$
$\chi_{\text{pcf}}^{n-1} \geq n^2 = n(\Delta - 1)$



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OPEN:

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Conjecture (2021.12+ Petruševski, Škrekovski)

If G is planar, then $\chi_{\mathrm{odd}}(G) \leq 5$. *Tight for 5-cycle.*

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"If G is planar with girth ≥ 6, then $\chi_{\text{odd}}(G) \le 4$."
implies Four Color Theorem!

Find min girth g such that
"If G is planar with girth $\ge g$, then $\chi_{\text{odd}}(G) \le c$ or $\chi_{\text{pcf}}(G) \le c$."

colors	3	4	5	6	7	8
χ_{odd}	x	11	7	5	4	all planar
χ_{pcf}	x	11	7	6	5	all planar

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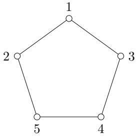
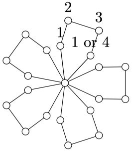
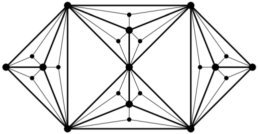
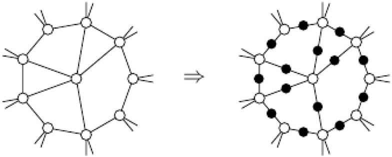
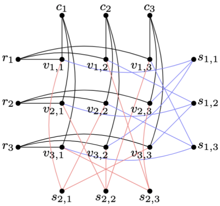






Thank you for your attention!





Theorem (2022.03+ Liu)

If G is planar or projective planar, then $\chi_{\mathrm{pcf}}(G) \leq 11$.
If G has Euler genus $\gamma \geq 2$, then $\chi_{\mathrm{pcf}}(G) \leq \frac{13+\sqrt{73+48\gamma}}{2}$.

Theorem (2022.05+ Metrebian)

If G is projective planar or toroidal, then $\chi_{\mathrm{odd}}(G) \leq 9$.

Theorem (2022.03+ Cranston, Lafferty, Song, 2022.06+ Liu, Wang, Yu)

If G is 1-planar, then $\chi_{\mathrm{odd}}(G) \leq 23$.
If G is 1-planar, then $\chi_{\mathrm{odd}}(G) \leq 13$.

Theorem (2022.02+ Dujmović,Morin,Odak, 2022.03+ Hickingbotham, 2022.03+ Liu)

If G is k-planar, then $\chi_{\mathrm{odd}}(G) \leq C \cdot k^5$.
If G is k-planar, then $\chi_{\mathrm{pcf}}(G) \leq 60k + 59$.
If G is k-planar and Euler genus γ, then $\chi_{\mathrm{pcf}}(G) \leq 16(\gamma+3)(k+1) - 1$.