Hyperfinite graphs and graphons

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Outline

- Extremal graph theory studies graphs that satisfy some extremal problems.
- If a graph with n vertices contains no triangle, then how many edges can it have at most?
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Turán problem

Theorem (Mantel, 1907)

If a graph with n vertices has more than $\frac{n^2}{4}$ edges, then it must contain a triangle.

Theorem (Turán, 1941)

If a graph with n vertices has more than $\frac{(k-2)n^2}{2(k-1)}$ edges, then it must contain a k-clique (a.k.a. k-complete graph).

Remark

Turán problem: Let G and H be two graphs. If G is H-free, then how many edges can G have at most?

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Erdős' work

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Szemerédi theorem

The upper density of $A \subseteq \mathbb{Z}$ is

$$\limsup_{n\to\infty} \frac{|A \cap \{1, 2, 3, \ldots, n\}|}{n}.$$

Theorem

Every subset of integers with positive upper density contains arbitrarily long arithmetic progressions.

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- Let G and H be two graphs. Let $\hom(H, G)$ denote the number of homomorphisms from H to G.
- The homomorphism density from H to G is

$$t(H, G) = \frac{\hom(H, G)}{|V(G)|^{|V(H)|}}.$$

- $t(K_2, G) \approx$ the edge density of G.
- Let C_4 denote 4-cycles, which is exactly $\square$.

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Motivation problem from extremal graph theory

Problem How many 4-cycles must a graph with edge density at least $\frac{1}{2}$ have?

Theorem (Erdős, 1938)

$$t(C_4, G) \geq t(K_2, G)^4.$$

In particular, if $t(K_2, G) \geq \frac{1}{2}$*, then* $t(C_4, G) \geq \frac{1}{16}$.

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(Extremal problem) Minimize $t(C_4, G)$ over all finite graphs satisfying $t(K_2, G) \geq \frac{1}{2}$.

- There exist graph sequences $\{G_n\}_{n\in\mathbb{N}}$ such that $t(K_2, G_n) \geq \frac{1}{2}$ and $t(C_4, G_n) \to \frac{1}{16}$.
- $\inf_G t(C_4, G) = \frac{1}{16}$.
- However, no finite graph with edge density $\geq \frac{1}{2}$ has 4-cycles density exactly $\frac{1}{16}$.
- The minimum will never be achieved.

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Graphons

- In 2006, Lovász and Szegedy introduced the notion of graphon, short for graph function.
- A graphon is a limit of a sequence of finite graphs as the vertices goes to infinity.
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Why a function from $[0,1]^2$ to $[0,1]$?

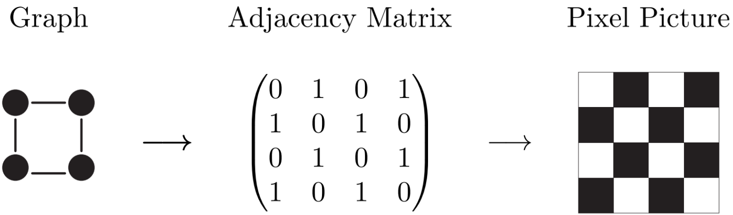

Figure: Source: D. Glasscock

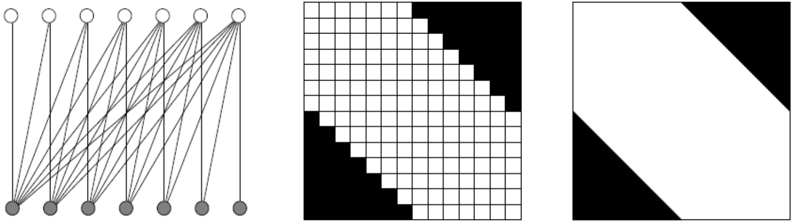

FIGURE 1.7. A half-graph, its pixel picture, and the limit function

Figure: Source: L. Lovász

The homomorphism density of a graphon

- Let $W\colon [0,1]^2 \to [0,1]$ be a symmetric Lebesgue measurable function, i.e., a graphon.
- The edge density of W is defined as

$$t(K_2, W) = \int_{[0,1]^2} W(x,y)dxdy.$$

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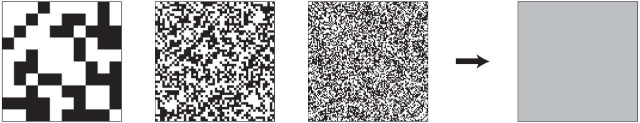
Random graphs

Let R_n denote a random graph with edge density $\frac{1}{2}$.



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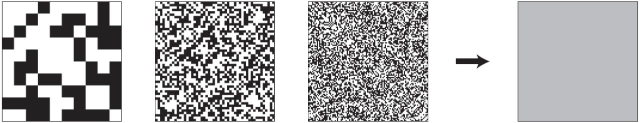

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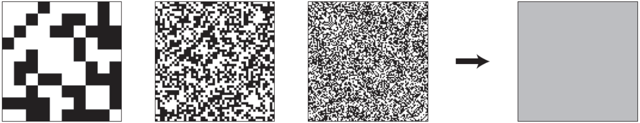

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Internal sets

- An object A in the nonstandard universe $V({}^*\mathbb{X})$ is an *internal set* if there is $B \in V(\mathbb{X})$ such that $A \in {}^*B$.
- A subset of ${}^*\mathbb{X}$ is an *external set* if it's not internal.
- Boolean combinations of internal sets are internal.
- Every standard element is internal.
- Let $f\colon \mathbb{X} \to \mathbb{X}$ be a function and let A be an internal set. Then ${}^*f(A)$ and $({}^*f)^{-1}(A)$ are internal.

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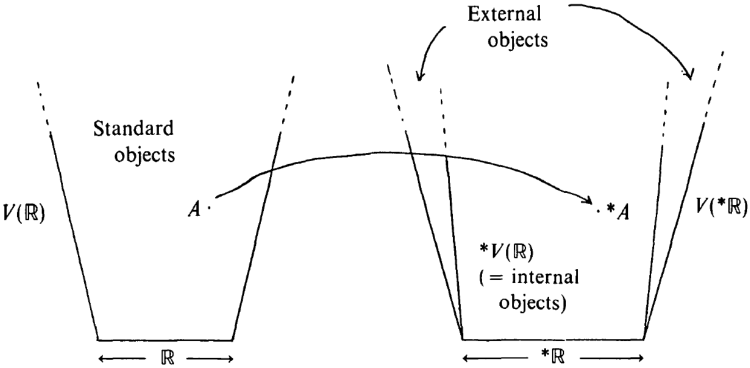

Figure: Source: N. Cutland

Hyperfinite sets

An internal set A is *hyperfinite* if there is an internal bijection $f\colon \{1, 2, \cdots, H\} \to A$ for some $H \in {}^*\mathbb{N}$. In fact, this H is unique. We use $|A|$ to denote the internal cardinality H of A.

Hyperfinite sets in nonstandard extensions correspond to finite sets in standard math.

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Internal finitely additive measure space

A triple $(\Omega, \mathcal{A}, P)$ is an internal finitely additive measure space if it satisfies

1. Ω is internal
2. $\mathcal{A}$ is an internal subalgebra on Ω
3. $P\colon \mathcal{A} \to {}^*\mathbb{R}$ is an internal function that satisfies:
   (a) $P(\emptyset) = 0$;
   (b) $P(\Omega) = 1$;
   (c) P is finitely additive, i.e.,
   $P(A \cap B) = P(A) + P(B) - P(A \cap B)$.

Remark

- *zero probability event and infinitesimal probability event are different.*
- $\mathrm{st} \circ P$ *is not countably additive.*

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Loeb theorem

Theorem (Loeb, 1975)

Let $(\Omega, \mathcal{A}, P)$ be an internal finitely additive probability space. Then there is a (standard) countably additive probability space $(\Omega, \mathcal{A}_L, P_L)$ satisfying

- $\mathcal{A}_L \supseteq \mathcal{A}$ *is a* σ*-algebra*
- $P_L = \mathrm{st} \circ P$ *in* $\mathcal{A}$
- *for every* $A \in \mathcal{A}_L$ *and positive* $\epsilon \in \mathbb{R}$*, there are* $B, C \in \mathcal{A}$ *such* $B \subseteq A \subseteq C$ *and* $P(C \setminus B) < \epsilon$
- *for every* $A \in \mathcal{A}_L$*, there is* $B \in \mathcal{A}$ *such that* $P_L(A \triangle B) = 0$

We call $(\Omega, \mathcal{A}_L, P_L)$ *Loeb probability space.*

Nonstandard construction of Lebesgue measure

- Take $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, consider $\Omega = \{0, \frac{1}{H}, \frac{2}{H}, \cdots, \frac{H-1}{H}\}$.
- Ω is a hyperfinite space. Let $\mathcal{A}$ be the collection of internal subsets of Ω. For every $A \in \mathcal{A}$, define $\nu(A) = \frac{|A|}{|H|}$, where $|\cdot|$ is internal cardinality. We call ν the counting measure on Ω.
- $(\Omega, \mathcal{A}, \nu)$ is an internal finitely additive probability space.
- By Loeb Theorem, we get the uniform hyperfinite Loeb probability space $(\Omega, \mathcal{A}_L, \nu_L)$.
- push-forward measure $\mathrm{st}_*(\nu_L) = \lambda$ (Lebesgue measure).

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Measure algebras

- A measure space is a triple $(X, \mathcal{A}, \mu)$ where X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X, and $\mu\colon \mathcal{A} \to [0, \infty)$ is a countably additive finite-valued measure.
- A measured algebra is a pair $(\mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-complete boolean algebra and $\mu\colon \mathcal{A} \to [0, \infty)$ is a finite-valued function such that $\mu(a) = 0$ iff $a = \mathbf{0}$, and μ is countably additive.
- A boolean algebra $\mathcal{A}$ is said to be a measure algebra if there is a finite-valued μ for which $(\mathcal{A}, \mu)$ is a measured algebra. A measured algebra (A, μ) is called a probability algebra if $\mu(\mathbf{1}) = 1$.

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- A measured algebra is a pair $(\mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-complete boolean algebra and $\mu \colon \mathcal{A} \to [0, \infty)$ is a finite-valued function such that $\mu(a) = 0$ iff $a = \mathbf{0}$, and μ is countably additive.
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Measure algebras as metric spaces

Let $(\mathcal{A}, \mu)$ be a measured algebra. For all $a, b \in \mathcal{A}$, define

$$d(a, b) = \mu(a \triangle b),$$

where $\triangle$ is the symmetric difference of those two sets. It is shown that $(\mathcal{A}, d)$ is a complete metric space.

Probability algebras associated to probability spaces

- Let $(X, \mathcal{A}, \mu)$ be a measure space. For all $a, b \in \mathcal{A}$, we write $a \equiv_\mu b$ if $\mu(a \triangle b) = 0$. Note that $\equiv_\mu$ defines an equivalence relation, and the equivalence class of a under $\equiv_\mu$ is denoted by $[a]_\mu$.
- Let $\widehat{A}$ denote the set $\{[a]_\mu \mid a \in \mathcal{A}\}$. Naturally, $\widehat{A}$ is a σ-complete boolean algebra.
- Moreover, μ induces a countably additive, strictly positive measure on $\widehat{A}$. We call $(\widehat{A}, \mu)$ the measured algebra associated to $(X, \mathcal{A}, \mu)$ and we call $\widehat{A}$ the measure algebra associated to $(X, \mathcal{A}, \mu)$.
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Maharam's Theorem

Theorem (Maharam's Theorem)

For all atomless probability spaces Ω, there is a countable set of distinct infinite cardinals $S = \{\kappa_i \mid i \in I\}$ such that the measure algebra of Ω is isomorphic to a convex combination of the homogeneous probability algebras $[0, 1]^{\kappa_i}$. The set S is uniquely determined by Ω and is called the Maharam spectra of Ω. □

Remark

The Maharam spectra of $[0, 1]^{\kappa}$ is $\{\kappa\}$, where κ is infinite.

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Maharam spectra of hyperfinite Loeb spaces

Theorem (Jin and Keisler)

Let $(\Omega, \mathcal{A}, \mu)$ *be a hyperfinite set with the normalized counting probability measure. Then the Maharam spectra of its corresponding Loeb space is* $\{\mathrm{Card}(2^{|\Omega|})\}$.

R. Jin and H. J. Keisler, *Maharam spectra of Loeb spaces*, J. Symb. Log. 65 (2000), 550–566.

Atomlessness

- A measure space $(X, \mathcal{A}, \mu)$ is atomless if for every $a \in \mathcal{A}$ with $\mu(a) > 0$, there exists $b \in \mathcal{A}$ such that $b \subseteq a$ and $0 < \mu(b) < \mu(a)$.
- Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{B}$ be a σ-subalgebra of $\mathcal{A}$. We say that $\mathcal{A}$ is atomless over $\mathcal{B}$, if for every $a \in \mathcal{A}$ of positive measure, there exists $b \in \mathcal{A}$ such that for all $c \in \mathcal{B}$, we have $a \cap b \neq a \cap c$.

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- Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{B}$ be a σ-subalgebra of $\mathcal{A}$. Given an infinite cardinal κ, we say that $\mathcal{A}$ is κ-atomless over $\mathcal{B}$, if for every σ-subalgebra $\mathcal{B}'$, which is σ-generated by $\mathcal{B} \cup S$, where S is a set of cardinality $< \kappa$ in $\mathcal{A}$, we have that $\mathcal{A}$ is atomless over $\mathcal{B}'$.

- When $\mathcal{B}$ is trivial and $\mathcal{A}$ is κ-atomless over $\mathcal{B}$, we say simply that $(X, \mathcal{A}, \mu)$ is κ-atomless.

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Maharam's Lemma

Lemma (Maharam's Lemma)

Let $(X, \mathcal{A}, \mu) \supseteq (X, \mathcal{B}, \mu)$ be measure spaces. Then the following are equivalent:

1. *The measure space $(X, \mathcal{A}, \mu)$ is atomless over $(X, \mathcal{B}, \mu)$.*
2. *For every $a \in \mathcal{A}$ of positive measure and for every $\mathcal{B}$-measurable function $f : X \to \mathbb{R}$ such that $0 \leq f \leq \mathbb{E}(a \mid \mathcal{B})$, there is a set $b \in \mathcal{A}$ such that $b \subseteq a$ and $\mathbb{E}(b \mid \mathcal{B}) = f$.*

Maharam spectra and atomlessness

Theorem (S.)

If Ω is an atomless probability space and κ is an infinite cardinal, then the following are equivalent:

1. *Ω is κ-saturated.*
2. *Every cardinal in the Maharam spectra of Ω is $\geq \kappa$.*
3. *Ω is κ-atomless.*

Hyperfinite graphs

- Take a hyperfinite set $N = \{1, 2, \cdots, H\}$.
- Consider internal graph G on N, i.e., a graph whose edge set $E(G) \subseteq N \times N$ is an internal subset.
- A hyperfinite (internal) graph is a graph.

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Lebesgue sample

Theorem (Bernstein and Wattenberg)

There is a hyperfinite set S satisfying $[0,1] \subseteq S \subseteq {}^*[0,1]$ such that for every Lebesgue measurable set $B \subseteq [0,1]$,

$$\lambda(B) = \mathrm{st}\left(\frac{|{}^*B \cap S|}{|S|}\right).$$

Such S is called a Lebesgue sample of $[0,1]$.

A. R. Bernstein and F. Watternberg, *Nonstandard measure theory*, in: Applications of model theory to algebra, analysis and probability (Ed. W. A. J. Luxemburg, Holt, Rinehard and Winston), New York, 1969, 171–185.

Remark

For every Lebesgue measurable set $B \subseteq [0,1]$,

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The Lebesgue measure of Lebesgue measurable $B \subseteq [0,1]$ $\approx$ the density of the set of elements in Lebesgue sample S that is infinitely close to B in S.

Main Theorem

Theorem (S.)

Let S be a Lebesgue sample of $[0,1]$ *and* $(S, L(S), \nu_L)$ *the uniform Loeb probability space on S. Let* $(S \times S, L(S \times S), (\nu \times \nu)_L)$ *be the product Loeb probability space. Consider the standard part map* $\mathrm{st} \times \mathrm{st} \colon S \times S \to [0,1]^2$*. Let $\mathcal{C}$ denote* $(\mathrm{st} \times \mathrm{st})^{-1}(\mathcal{L}([0,1]^2))$*. Then, for every graphon* $f \colon [0,1]^2 \to [0,1]$*, there is an internal graph G on S such that* $f = \mathbb{E}(E(G)|\mathcal{C})|_{[0,1]\times[0,1]}$.

Sketch of the proof I

- Let $L_0(S) = \mathrm{st}^{-1}(\mathcal{L}([0,1]))$. Then,

$$\mathrm{st}\colon (S, L_0(S), \nu_L) \to ([0,1], \mathcal{L}([0,1]), \lambda)$$

is an isomorphism between probability spaces.

- The Maharam spectrum of $(S, L_0(S), \nu_L)$ is $\{\aleph_0\}$.
- By Jin and Keisler's theorem, the Maharam spectrum of $(S, L(S), \nu_L)$ is $\mathrm{Card}(2^{|S|}) \geq 2^{2^{\aleph_0}} > 2^{\aleph_0}$.
- Then $(S, L(S), \nu_L)$ is $2^{2^{\aleph_0}}$-atomless, and thus $L(S)$ is atomless over $L_0(S)$.

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Sketch of the proof II

- Consider product Loeb space $(S \times S, L(S \times S), (\nu \times \nu)_L)$. Anderson showed that $\overline{L(S) \times L(S)} \subsetneq L(S \times S)$ and $\nu_L \times \nu_L \subseteq (\nu \times \nu)_L$.
- Since $L(S)$ is atomless over $\overline{L_0(S)}$, it follows that $\overline{L(S) \times L(S)}$ is atomless over $\overline{L_0(S) \times L_0(S)}$, and thus, $L(S \times S)$ is atomless over $\overline{L_0(S) \times L_0(S)}$.
- The mapping from $L^1(([0,1]^2, \mathcal{L}([0,1]^2), \lambda \times \lambda), [0,1])$ to $L^1((S \times S, \overline{L_0(S) \times L_0(S)}, \nu_L \times \nu_L), [0,1])$ by sending f to $(\mathrm{st} \times \mathrm{st}) \circ f$ is isomorphic between random variable structures. Let $\mathcal{C}$ denote $\overline{L_0(S) \times L_0(S)}$.

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Sketch of the proof III

- Since $L(S \times S)$ is atomless over $\mathcal{C}$, by Maharam's Lemma and Lifting Theorem for every Lebesgue measurable $f\colon [0,1] \times [0,1] \to [0,1]$ there is a Loeb measurable set $A \subseteq S \times S$ such that $\mathbb{E}(A|\mathcal{C}) = (\mathrm{st} \times \mathrm{st}) \circ f$.
- Then by Loeb Theorem, there is an internal set $B \subseteq S \times S$ satisfying $(\nu \times \nu)_L(A \triangle B) = 0$ such that $\mathbb{E}(B|\mathcal{C}) = (\mathrm{st} \times \mathrm{st}) \circ f$.
- If $f\colon [0,1] \times [0,1] \to [0,1]$ is a graphon, then we get an internal graph G on S such that $\mathbb{E}(E(G) \mid \mathcal{C})_{[0,1]\times[0,1]} = f$.

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Thanks

Thank you for your attention !