# Small Quasi-Kernel

## for Claw-free and One-way Split Digraphs

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# Content

# Kernels and Quasi-Kernels

**Definition**

A kernel is an independent set $K \subseteq V$ such that for any vertex $v \in V \setminus K$ there exists a directed path with one arc from $v$ to a vertex $u \in K$.

**Definition**

A quasi-kernel is an independent set such that for any vertex $v \in V \setminus Q$, there exists a directed path with at most two arcs from $v$ to a vertex $u \in Q$.

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*Every digraph has a quasi-kernel.*

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- If $N^+(v) \cup Q' \neq \emptyset$, $Q$ is a quasi-kernel of $D$.

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- Let $Q'$ be a quasi-kernel of $D'$.
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- Otherwise, $Q' + v$ is a quasi-kernel of $D$.

□

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**Theorem**

Every digraph without odd cycles has a kernel.

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**Conjecture 2([A. Kostochka, R. Luo, and S. Shan, 2020])**

Let $D$ be an $n$-vertex digraph, and let $S$ be the set of sinks of $D$. Then $D$ has a quasi-kernel $Q$ such that $|Q| \leq \frac{n+|S|-|N^-(S)|}{2}$.

# Our Result I (Anti-Claw-Free Digraph)

**theorem 1**

Every sink-free digraph with no induced $\vec{K}_{4,1}$ and no induced $\vec{K}_{4,1}^{+}$ has a small quasi-kernel.

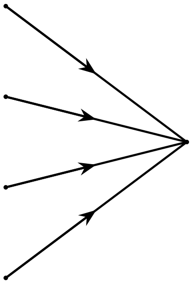


Figure: $\vec{K}_{4,1}$

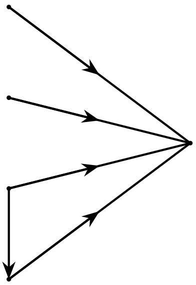


Figure: $\vec{K}_{4,1}^{+}$

theorem 2

Every sink-free digraph with no anti-claw has a small quasi-kernel.

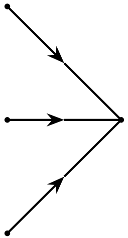
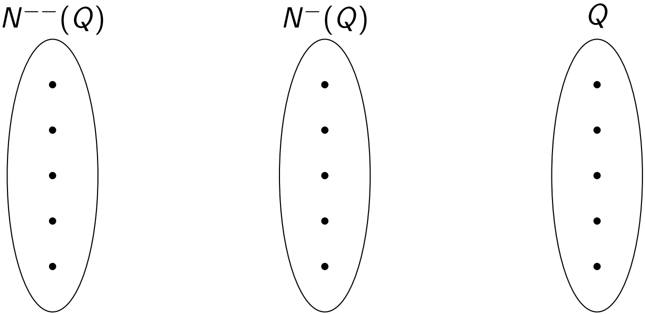
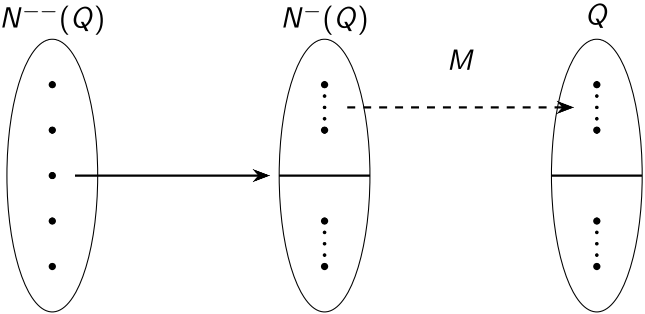


Figure: $\vec{K}_{3,1}$

# Quasi-kernel $Q$ and its second neighbourhood



# The Maximal Matching From $N^{-}(Q)$ to $Q$



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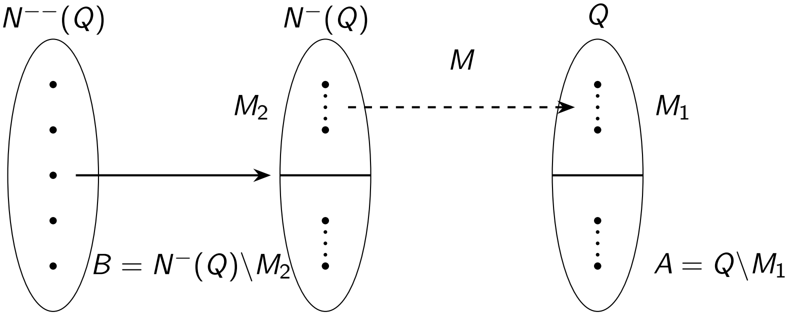

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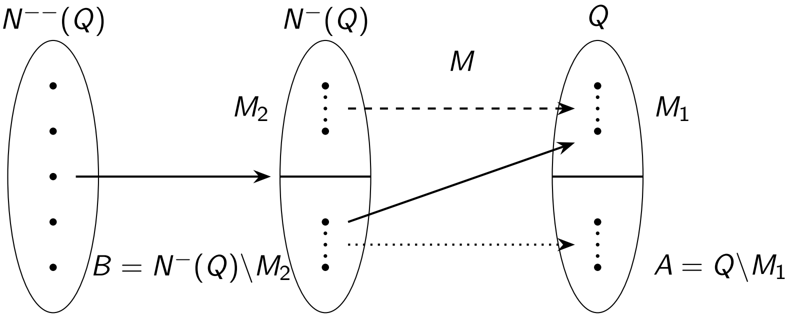

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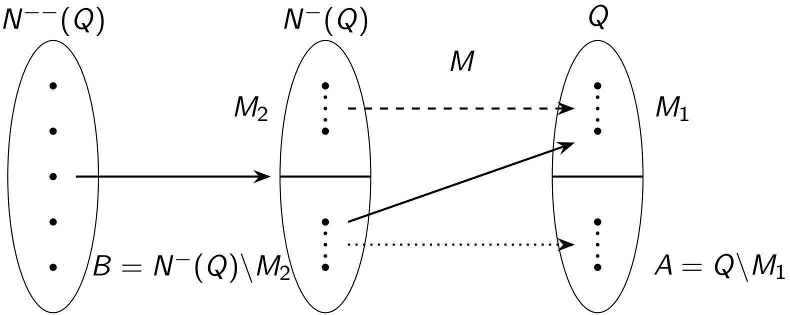
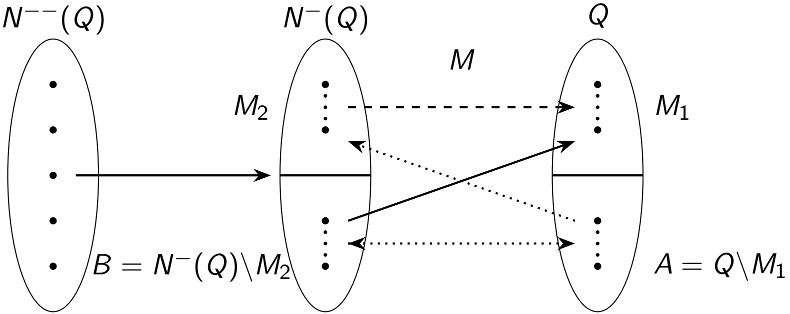



**Obervation 1**

$M_1$ is a quasi-kernel of $D[V \setminus A]$.

# The Maximal Matching From $N^-(Q)$ to $Q$

For a minimal quasi-kernel $Q$ in $D$




## Observation 2

If $Q$ is minimal quasi-kernel then there is no arc from $A$ to $N^-(Q)$.

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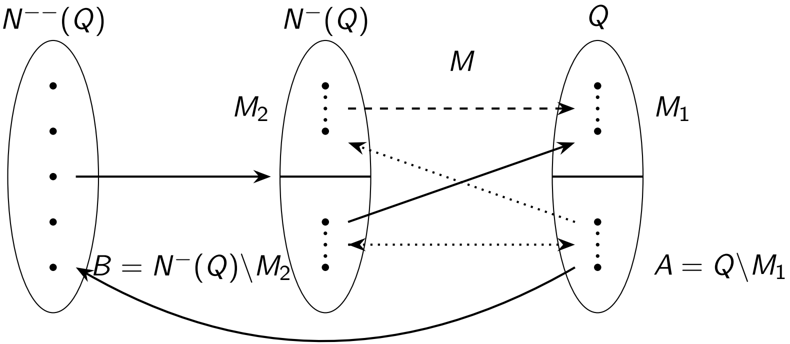


## Observation 3

If $Q$ is minimal quasi-kernel then for all $v \in A$, $|N^{+}(v) \cap N^{--}(Q)| \geq 1$.

# Some Obervations of the Structures

If $Q$ is a good quasi-kernel, that is, a quasi-kernel $Q$ such that every vertex $v \in Q$ has an arc to $N^-(Q)$ then we can remove all vertices in $A$ to obtain a quasi-kernel that is no larger than its in-neighbourhood and thus is small.

**Observation 4**

If a digraph $D$ has a good quasi-kernel then $D$ has a small quasi-kernel.

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**Observation 4**

If a digraph $D$ has a good quasi-kernel then $D$ has a small quasi-kernel.

If a digraph with a kernel is sink-free then every kernel is a good quasi-kernel. Thus we obtain the following observation first proved by van Hulst [A. van Hulst, 2021].

**Observation 5**

Every sink-free digraph with a kernel has a small quasi-kernel.

# Sketch of the Proof

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2. By Observation 3, $N^{+}(A) \subseteq N^{--}(Q)$. Therefore, there must exist a vertex $v \in N^{--}(Q)$ that has (at least) two in-neighbours in $A$.
3. $v$ also must have an in-nerghbour in $M_1$ which together with 2 in-neighbours in $A$ forms an anti-claw, a contradiction.

## Observation 6

Let $Q$ be a quasi-kernel of a digraph $D$, and let $\tilde{Q}$ be a quasi-kernel of $D[N^{--}(Q)]$. Then $(Q \cup \tilde{Q}) \setminus N^{-}(\tilde{Q})$ is a quasi-kernel of $D$.

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## Lemma 1

Let $D$ be a sink-free digraph. If $D$ has a quasi-kernel $Q$ such that $D[N^{--}(Q)]$ has a kernel, then $D$ has a small quasi-kernel.

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Let $D$ be an $n$-vertex digraph and $S$ be the set of sinks of $D$. Suppose that $V(D)\setminus N^{-}[S]$ has a partition $V_1 \cup V_2$ such that $D[V_i]$ is kernel-perfect for each $i = 1, 2$. Then D has a quasi-kernel of size at most $(n + |S| - |N^{-}(S)|)/2$.

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A digraph is kernel-perfect if every induced subdigraph of it has a kernel. Note that every digraph with no odd cycle is kernel-perfect. In particular, bipartite digraphs are kernel-perfect.

# The One-way Split Digraphs

**Definition**

A digraph $D$ is called a one-way split digraph, if its vertex set can be partitioned into $X$ and $Y$, such that $X$ induces an independent set and $Y$ induces a semicomplete digraph (a digraph in which there is at least one arc between every pair of vertices) and any arcs between $X$ and $Y$ go from $X$ to $Y$ .

# Our Result II (One-way Split Digraphs)

**Theorem**

*Let $D$ be an one-way split digraph of order $n$ with no sinks. Then $D$ has a quasi-kernel of size at most $\frac{n+3}{2} - \sqrt{n}$. Furthermore, for infinitely many values of $n$ there exists a one-way split digraph of order $n$, with no sink, such that the minimum size of quasi-kernels of $D$ is $\frac{n+3}{2} - \sqrt{n}$.*

## Sketch of the Proof

1. Let $D$ be a one-way split digraph of order $n$ with no sink. Let $X$ and $Y$ be a partition of $V(D)$ such that $X$ is independent and $Y$ induces a semicomplete digraph and all arcs between $X$ and $Y$ go from $X$ to $Y$.

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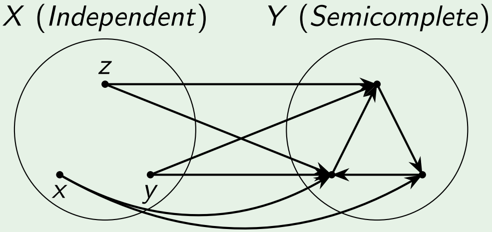

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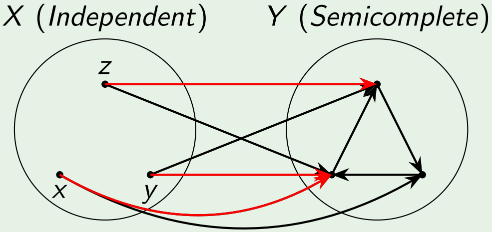

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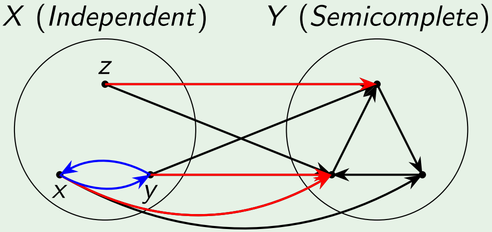

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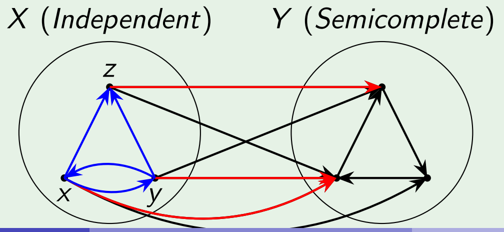

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4. Observe that there is at least one arc between any pair of vertices in $H$.

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4. 

$$|Q| \leq 1 + |X| - \left(\frac{|X| - 1}{2} + \frac{|X|/|Y| - 1}{2} + 1\right) = \frac{|X|}{2} - \frac{|X|/|Y|}{2} + 1$$

# A Extremal Example

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We let $k \geq 1$ be any integer and construct the digraph $D_k$ of order $(2k+1)^2$ as follows. Let $T$ be a $k$-regular tournament of order $2k+1$ and for each vertex, $v$, of $T$ add $2k$ new vertices, $V_v$, with arcs into $v$. The resulting digraph, $D_k$, has order $(2k+1)^2$ and is a one-way split digraph with partition $V(T)$ (the tournament) and $V(D_k) \setminus V(T)$ (the independent set).

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We let $k \geq 1$ be any integer and construct the digraph $D_k$ of order $(2k+1)^2$ as follows. Let $T$ be a $k$-regular tournament of order $2k+1$ and for each vertex, $v$, of $T$ add $2k$ new vertices, $V_v$, with arcs into $v$. The resulting digraph, $D_k$, has order $(2k+1)^2$ and is a one-way split digraph with partition $V(T)$ (the tournament) and $V(D_k) \setminus V(T)$ (the independent set).

The size of quasi-kernels in $|Q|$:

$$|Q| \geq 2k^2 + 1 = \frac{4k^2 + 4k + 1}{2} - \frac{4k + 2}{2} + \frac{3}{2} = \frac{n}{2} - \sqrt{n} + \frac{3}{2}$$

Thank you for your attention!

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