Thresholds

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SCMS
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New result

Conjecture [Kahn-Kalai '06]; proved by P.-Pham ('22).

There exists a universal $K > 0$ such that for every finite set X and increasing property $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq K p_{\mathbf{E}}(\mathcal{F}) \log |X|$$

- $p_c(\mathcal{F})$: *threshold* for $\mathcal{F}$
- $p_{\mathbf{E}}(\mathcal{F})$: *expectation threshold* for $\mathcal{F}$

Basic definitions

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  e.g.1. $X = \binom{[n]}{2} = E(K_n)$

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- $\mathcal{F} \subseteq 2^X$ is an **increasing property** if

$$B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$$

  e.g.1. $\mathcal{F} = \{\text{connected}\}$; $\mathcal{F} = \{\text{contain a triangle}\}$

  e.g.2. $\mathcal{F} = \{\text{not satisfiable}\}$

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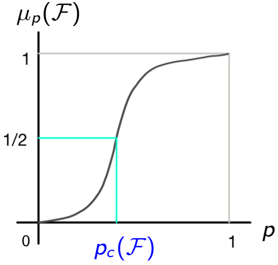
Fact.

For any increasing property $\mathcal{F}$ ($\neq \emptyset, 2^X$), $\mu_p(\mathcal{F})$ ($= \mathbb{P}(X_p \in \mathcal{F})$) is continuous and strictly increasing in p.



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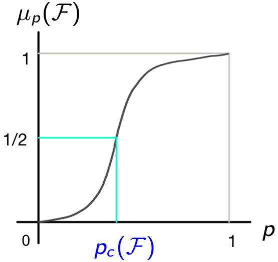

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- cf. Erdős-Rényi: $p_0 = p_0(n)$ is **a threshold function** for $\mathcal{F}_n$ if

$$\mu_p(\mathcal{F}_n) \to \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases}$$

* $p_c(\mathcal{F}_n)$ is always an Erdős-Rényi threshold (Bollobás-Thomason '87).

The Kahn–Kalai Conjecture

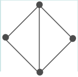
"It would probably be more sensible to conjecture that it is **not** *true."*

- Kahn and Kalai (2006)

Question.

What drives $p_c(\mathcal{F})$?

Example 1. Containing a copy of H



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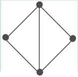
Example 1. Containing a copy of H 

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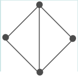

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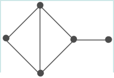
- triv. $p_c(\mathcal{F}_H) \gtrsim n^{-4/5}$ $(\because \mathbb{E}X \to 0 \Rightarrow X = 0$ with high probability$)$
- truth: $p_c(\mathcal{F}_H) \asymp n^{-4/5}$

Example 2. Containing a copy of K



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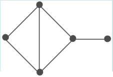
Example 2. Containing a copy of K 

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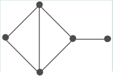
Example 2. Containing a copy of K 

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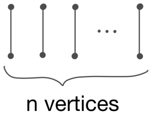
Erdős-Rényi ('60), Bollobás ('81)

(Rough:) For **fixed** graph K,

$$p_c(\mathcal{F}_K) \asymp \text{"threshold for } \mathbb{E}\text{" of the "densest" subgraph of K}$$

Example 3. Containing a perfect matching



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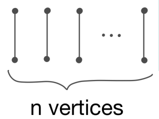

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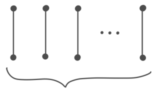

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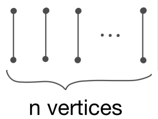

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- truth: $p_c(\mathcal{F}) \asymp \log n/n$ ← **another trivial lower bound**

Fact. $p \ll \log n/n \Rightarrow G_{n,p}$ has an isolated vertex w.h.p.

One more example: perfect hypergraph matchings

- Now, $X = \binom{[n]}{r}$
- $X_p =$ random r-uniform hypergraph $\mathcal{H}^r_{n,p}$

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- $p_c(\mathcal{F}) \asymp \log n / n^2$ (Johansson-Kahn-Vu '08) * $\log n$ **gap again**

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- We have some **trivial lower bounds** on p_c:
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$$p_{\mathbb{E}} \asymp n^{-(r-1)} \xRightarrow{\text{KKC}} p_c \lesssim \log n / n^{r-1}$$

  - threshold for bounded degree spanning trees ("tree conjecture"; Montgomery ’19)

$p_{\mathsf{E}}(\mathcal{F})$: the expectation threshold

- For abstract $\mathcal{F}$, it's unclear whose expectation we want to compute, so need a careful definition for the "threshold for $\mathbb{E}$."

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Observation

$p_c(\mathcal{F}) \geq q$ if $\exists\ \mathcal{G} \subseteq 2^X$ such that

1. "$\mathcal{G}$ covers $\mathcal{F}$": $\forall A \in \mathcal{F}\ \ \exists B \in \mathcal{G}$ such that $A \supseteq B$ $\quad (\mathcal{F} \subseteq \langle \mathcal{G} \rangle)$
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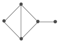
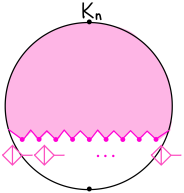
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of K 



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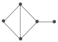
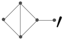
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of K 

- $\mathcal{G}_1 = \{$all (labeled) copies of K  $'s\}$

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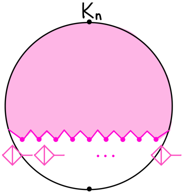

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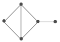
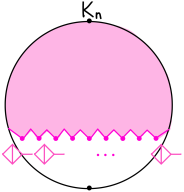
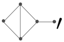
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of K 



- $\mathcal{G}_1 = \{$all (labeled) copies of K  $'s\}$

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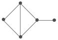
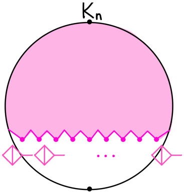
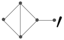
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of K 



- $\mathcal{G}_1 = \{$all (labeled) copies of K  $'s\}$

  $\rightarrow \sum_{S \in \mathcal{G}_1} q^{|S|} \leq 1/2$ for $q \lesssim n^{-5/6}$ $\rightarrow n^{-5/6} \lesssim p_c(\mathcal{F})$

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$p_E(\mathcal{F})$: the expectation threshold

Observation

$p_c(\mathcal{F}) \geq q$ if $\exists\, \mathcal{G} \subseteq 2^X$ such that

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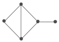
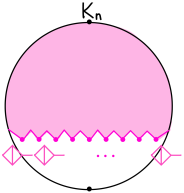
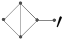
e.g. in Ex 2, $X = \binom{[n]}{2}$, $\mathcal{F}$: contain a copy of K 



- $\mathcal{G}_1 = \{$all (labeled) copies of K  $'s\}$

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The Kahn-Kalai Conjecture ('06)

There exists a universal $K > 0$ such that for every finite X and increasing $\mathcal{F} \subseteq 2^X$,

$$(p_{\mathsf{E}}(\mathcal{F}) \leq)\ p_c(\mathcal{F}) \leq K p_{\mathsf{E}}(\mathcal{F}) \log |X|$$

Results and Proof Sketch

Conj of Talagrand: fractional version of Kahn-Kalai Conj

- $p^*_{\mathsf{E}}(\mathcal{F})$: the **fractional expectation threshold** for $\mathcal{F}$
  - skip def: roughly, replace cover $\mathcal{G}$ by "fractional cover"

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- Weaker than KKC, but in all known applications, $p_{\mathsf{E}}(\mathcal{F}) \asymp p^*_{\mathsf{E}}(\mathcal{F})$
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"Erdős-Rado Sunflower Conjecture"

$p_{\mathsf{E}}(\mathcal{F})$ vs. $p^*_{\mathsf{E}}(\mathcal{F})$

FKNP (19') $p_c(\mathcal{F}) \le K p^*_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$

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- Implies equivalence of KKC and fractional KKC
  – the most likely way to prove KKC?
- Even simple instances of the conjecture are not easy to establish; Talagrand suggested two test cases, proved by (respectively) DeMarco-Kahn ('15) and Frankston-Kahn-P. ('21)

New result

Conjecture (Kahn-Kalai '06); proved by P.-Pham ('22)

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- Proofs inspired by ALWZ (sunflower) and FKNP (fractional Kahn-Kalai) but implementation very different
- Reformulation – think: $\mathcal{H} = \{\text{minimal elements of } \mathcal{F}\}$

Theorem (P.-Pham '22)

$\exists L > 0$ such that $\forall \ell$-bdd $\mathcal{H}$, if $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$, then, with $m = Lp \log \ell |X|$,

$$\mathbb{P}(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1)$$

Proof sketch

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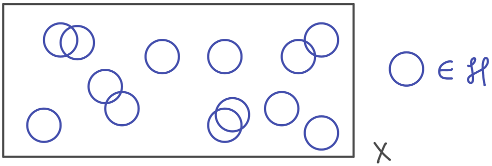

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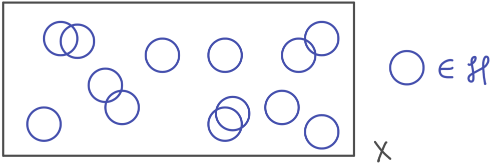

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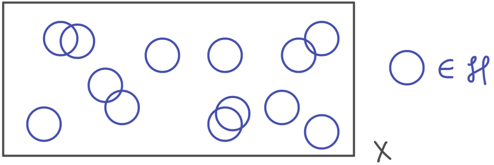

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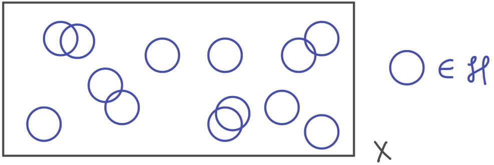

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- At the end, want $W \supseteq S \in \mathcal{H}$ whp.
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- (Recall) $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$ means:

$\langle \mathcal{H} \rangle$ does not admit a p-cheap cover.

$\exists L > 0$ such that $\forall \ell$-bdd $\mathcal{H}$, if $p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)$, then, with $m = Lp \log \ell |X|$,

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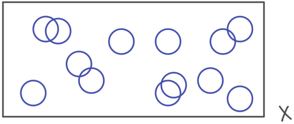

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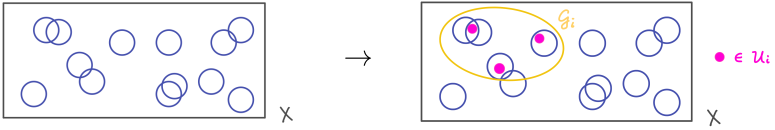

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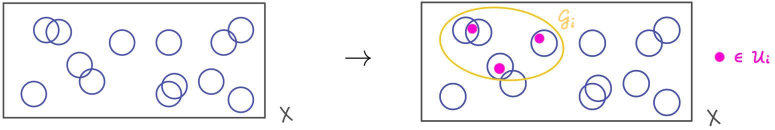

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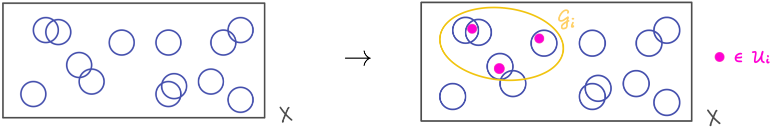

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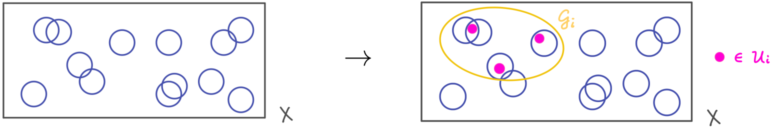

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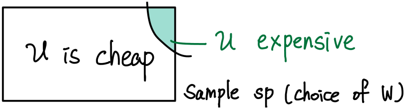

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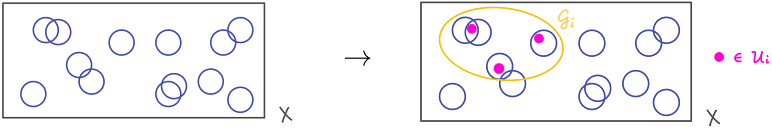

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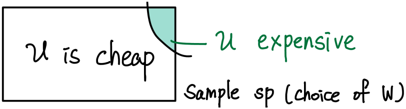

$\xrightarrow{\text{Ass. } p > p_{\mathsf{E}}(\langle \mathcal{H} \rangle)}$ (1) is unlikely $\square$

Open Questions

Gap between $p_{\mathsf{E}}(\mathcal{F})$ and $p_c(\mathcal{F})$

Theorem (P.-Pham '22)

$$(p_{\mathsf{E}}(\mathcal{F}) \leq)\; p_c(\mathcal{F}) \lesssim p_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$$

Question

What characterizes the gap between $p_{\mathsf{E}}(\mathcal{F})$ and $p_c(\mathcal{F})$?

Gap between $p_E(\mathcal{F})$ and $p_c(\mathcal{F})$

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Question

What characterizes the gap between $p_E(\mathcal{F})$ and $p_c(\mathcal{F})$?

- In many cases the $\log \ell(\mathcal{F})$ gap is tight:

  e.g. perfect hypergraph matchings, spanning trees with bounded degree, Hamiltonian cycle, fixed subgraphs...

- There are some cases for which $\log \ell(\mathcal{F})$ is not tight:

  e.g. clique factors, the k-th power of a Hamilton cycle, non-spanning large graphs... $\rightarrow$ good test cases!

Test cases: gaps smaller than $\log \ell(\mathcal{F})$ Thm. $p_c(\mathcal{F}) \leq K p_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$

First successful test case

$\mathcal{F}$: contain **the square of a Hamilton cycle (HC^2)**

Conjecture (Kühn-Osthus '12)

$$p_c(\mathcal{F}) \asymp n^{-1/2}$$

- $p_{\mathsf{E}}(\mathcal{F}) (\asymp p_{\mathsf{E}}^*(\mathcal{F})) \asymp n^{-1/2}$ $\rightarrow$ no gap!

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- $p_{\mathrm{E}}(\mathcal{F})(\asymp p^*_{\mathrm{E}}(\mathcal{F})) \asymp n^{-1/2}$ $\rightarrow$ no gap!
  - Kühn-Osthus ('12) $p^* \lesssim n^{-1/2+o(1)}$
  - Nenadov-Škorić ('16) $p^* \lesssim n^{-1/2} \log^4 n$
  - Fischer-Škorić-Steger-Trujić ('18) $p^* \lesssim n^{-1/2} \log^3 n$
  - Montgomery $p^* \lesssim n^{-1/2} \log^2 n$
  - Frankston-Kahn-Narayanan-P. $p^* \lesssim n^{-1/2} \log n$

Test cases: gaps smaller than $\log \ell(\mathcal{F})$ Thm. $p_c(\mathcal{F}) \le K p_{\mathsf{E}}(\mathcal{F}) \log \ell(\mathcal{F})$

First successful test case

$\mathcal{F}$: contain **the square of a Hamilton cycle (HC^2)**

Conjecture (Kühn-Osthus '12)

$$p_c(\mathcal{F}) \asymp n^{-1/2}$$

- $p_{\mathsf{E}}(\mathcal{F}) (\asymp p^*_{\mathsf{E}}(\mathcal{F})) \asymp n^{-1/2}$ $\rightarrow$ no gap!
    - Kühn-Osthus ('12) $p^* \lesssim n^{-1/2+o(1)}$
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Kahn-Narayanan-P. ('20)

$$p_c(\mathcal{F}) \asymp n^{-1/2}$$

Good test cases: gaps smaller than $\log \ell(\mathcal{F})$

[Ex 1] $\mathcal{F}$: contain a **triangle-factor** (or a H-factor for fixed H)

Johansson-Kahn-Vu ('08)

$$p_c(\mathcal{F}) \asymp n^{-2/3}(\log n)^{1/3}$$

Good test cases: gaps smaller than $\log \ell(\mathcal{F})$

[Ex 1] $\mathcal{F}$: contain a **triangle-factor** (or a H-factor for fixed H)

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[Ex 2] Perfect matchings in the "k-out model"

Good test cases: gaps smaller than $\log \ell(\mathcal{F})$

[Ex 1] $\mathcal{F}$: contain a **triangle-factor** (or a H-factor for fixed H)

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[Ex 2] Perfect matchings in the "k-out model"

Frieze ('86)

$$\lim_{\substack{n \to \infty \\ n \text{ even}}} \mathbb{P}(G_{k\text{-out}} \text{ has a perfect matching}) = \begin{cases} 0 & \text{if } k = 1 \\ 1 & \text{if } k \geq 2 \end{cases}$$

Thank you!