A Ramsey–Turán theory for tilings in graphs

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H-tiling

Given graphs H and G, an H-tiling is a collection of vertex-disjoint copies of H in G. An H-tiling is perfect (H-factor) if it covers all the vertices of G.

- (Dirac, '52) $\delta(G) \geq n/2 \Rightarrow G$ has a Hamilton cycle.
- (Hajnal–Szemerédi, '70) $\delta(G) \geq (1 - \frac{1}{k})n \Rightarrow G$ contains a K_k-factor. (Corrádi–Hajnal for $k = 3$)

Theorem (Alon–Yuster, '96, Komlós–Sárközy–Szemerédi, '98)

Given a graph H of k vertices, every graph G of n vertices with $n \in k\mathbb{N}$ large and $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n + C(H)$ has an H-factor.

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H-factor for general graph H

- (El-Zahar, '84) Let $n_1 + \cdots + n_k = n$ and let $\delta(G) \geq \sum_{i\in[k]} \lceil n_i/2 \rceil$. Then G contains k vertex-disjoint cycles of orders $n_1, \cdots, n_k$.
- (Komlós, '00) The critical chromatic number of H is

$$\chi_{cr}(H) := \frac{(r-1)k}{k-\sigma} = r - 1 + \frac{\sigma}{\frac{k-\sigma}{r-1}},$$

where $r = \chi(H)$ and $\sigma = \sigma(H)$ denotes the smallest size of a color class over all r-colorings of H. It is easy to see that

$$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H).$$

Theorem (Komlós, '00, Shokoufandeh–Zhao '03)

Given a graph H, every graph G on n vertices with n large and $\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n$ contains an H-tiling covering all but at most C vertices.

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H-factor

Theorem (Kühn–Osthus, '09)

For an arbitrary graph H, the relevant parameter for an H-factor is either $\chi_{cr}(H)$ or $\chi(H)$ and provide a dichotomy.

Ramsey-Turán theory

Erdős and Sós (1970) initiated a variation on Turán problem which excludes all graphs with large independence number.

- The ℓ-independence number of a graph G is denoted as

$$\alpha_\ell(G) := \max\{|S| : S \subseteq V(G), G[S] \text{ is } K_\ell\text{-free}\}.$$

- A central problem is to determine $\mathbf{RT}_\ell(n, H, o(n))$: the maximum number of edges in an n-vertex H-free graph G with $\alpha_\ell(G) = o(n)$.
- Define $\varrho_\ell(H) := \lim_{\alpha\to 0} \lim_{n\to\infty} \frac{\mathbf{RT}_\ell(n,H,\alpha n)}{\binom{n}{2}}$.
- (Bollobás–Erdős, '76, Szemerédi, '72) $\varrho_2(K_4) = \frac{1}{4}$.
- (Erdős–Hajnal–Sós–Szemerédi, 83) $\varrho_2(K_{2t}) = \frac{3t-5}{3t-2}$.
- $\varrho_2(K_{2,2,2})$?
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Clique factors

Problem A (Balogh–Molla–Sharifzadeh, '16)

Let $k \geq 3$ be an integer and G be an n-vertex graph with $\alpha(G) = o(n)$. What is the minimum degree condition on G that guarantees a K_k-factor?

- (Balogh–Molla–Sharifzadeh, '16) If $\delta(G) \geq \frac{n}{2} + o(n)$ and $\alpha(G) = o(n)$, then G contains a triangle-factor.
- (Knierim–Su, '20) For $k \geq 4$, if $\delta(G) = (1 - \frac{2}{k})n + o(n)$ and $\alpha_2(G) = o(n)$, then G contains a K_k-factor.

Question (Knierim–Su, '20)

Is it true that for every $k, \ell \in \mathbb{N}$ with $2 \leq \ell \leq k$, if G is an n-vertex graph with $\delta(G) = \max\{\frac{1}{2}n, (1 - \frac{\ell}{k})n\} + \Omega(n)$ and $\alpha_\ell(G) = o(n)$, then G contains a K_k-factor?

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We first give a negative answer for the interval $\frac{k}{2} \leq \ell \leq k - 3$.

Proposition (Chang–Han–Kim–Wang–Y, '21+)

Let $k, \ell \in \mathbb{N}$ such that $\frac{k}{2} \leq \ell \leq k - 3$. For any $\mu > 0$ and $\alpha > 0$ the following holds for sufficiently large $n \in k\mathbb{N}$. There exists an n-vertex graph G with $\delta(G) \geq \left(\frac{1}{2 - \varrho_\ell(K_{k-1})} - \mu\right) n$ and $\alpha_\ell(G) \leq \alpha n$ and G contains no K_k-factor.

- A recent result of Balogh and Lenz implies that $\varrho_\ell(K_{k-1}) > 0$ for any $\ell \leq k - 3$.

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Almost K_k-factor: the case $\ell = k - 1$

Theorem (Balogh–McDowell–Molla–Mycroft, '18)

For every constant $\mu > 0$ there are constants $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \mu n$ and $\alpha_2(G) \leq \alpha n$ contains a K_3-tiling covering all but at most 4 vertices in G.

Theorem (Han–Morris–Wang–Y, '21+)

For any integer $k \geq 3$ and constant μ, there exists a constant $\alpha > 0$ such that for any integer $r \in [2, k]$ and sufficiently large n, every n-vertex graph G with $\delta(G) \geq \frac{n}{r} + \mu n$ and $\alpha_{k-1}(G) \leq \alpha n$ contains a K_k-tiling that leaves at most $(k-1)(r-1)$ vertices uncovered. In particular, if $r = 2$ and $n \in k\mathbb{N}$, then G contains a K_k-factor.

- The proof uses the lattice-based absorbing method.
- What about the remaining cases $3 \leq \ell \leq k - 2$?

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- What about the remaining cases $3 \leq \ell \leq k - 2$?

Removing divisibility barriers

Recall that the construction of two or more vertex-disjoint cliques of almost equal size not divisible by k, has low independence number and essentially provides a barrier for F-factors.

Nenadov and Pehova suggest to strengthen the independence condition by forbidding large partite 'holes'.

Definition

For $k \in \mathbb{N}$ with $k \geq 2$, an k-partite hole of size s in a graph G is a collection of k disjoint vertex subsets $U_1, \dots, U_k \subset V(G)$ of size s such that there is no copy of K_k in G with exactly one vertex in each U_i, $i \in [k]$. We use $\alpha_k^*(G)$ to denote the size of the largest k-partite hole in G. When $k = 2$, we also refer to this as a bipartite hole and write $\alpha^*(G) = \alpha_2^*(G)$.

- It is clear from the definition that $k(\alpha_k^*(G) + 1) > \alpha_k(G)$.

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Clique factors revisted

Theorem (Nenadov–Pehova, 2020, Han–Morris–Wang–Y, '21+)

For any integer $k \geq 2$ and $\varepsilon > 0$, there exists a constant $\alpha > 0$ such that for large $n \in k\mathbb{N}$, every n-vertex graph G with $\delta(G) \geq \varepsilon n$ and $\alpha_k^*(G) \leq \alpha n$ contains a K_k-factor.

- The case $k = 2$ is of independent interest: a result of McDiarmid and Yolov implies that every graph G with $\delta(G) \geq 2\alpha_2^*(G)$ is Hamiltonian.

Triangle-factors under small bipartite hole

Proposition (Han–Morris–Wang–Y, 2021+)

For any $0 < \alpha < 1$, the following holds for sufficiently large integer $n \in 3\mathbb{N}$. There exists an n-vertex graph G with $\delta(G) \geq \frac{n}{2} - 2d^2$ and $\alpha^*(G) \leq \alpha n$ such that G contains no K_3-factor, where $d = \lceil (\frac{2}{\alpha} + 1)^2 \rceil$.

- A result of Balogh–Molla–Sharifzadeh states that $\delta(G) \geq \frac{n}{2} + o(n)$ and $\alpha(G) = o(n)$ force a K_3-factor.

Theorem (Ping–Hu–Wang–Wang–Y, 2022+)

For any integer Δ and $\varepsilon > 0$, there exists a constant $\alpha > 0$ such that for large $n \in \mathbb{N}$, every n-vertex graph G with $\delta(G) \geq \varepsilon n$ and $\alpha_2^*(G) \leq \alpha n$ is $\mathcal{T}(n, \Delta)$-universal.

- This strengthens a conjecture of Krivelevich, Kwan and Sudakov on the $\mathcal{T}(n, \Delta)$-universality in a *random perturbation* model.

Theorem (Han–Morris–Wang–Y, 2021+)

For any integer $k \geq 4$ and $\varepsilon > 0$, there exists a constant $\alpha > 0$ such that for large $n \in k\mathbb{N}$, every n-vertex graph G with $\delta(G) \geq \frac{n}{k} + \varepsilon n$ and $\alpha_2^*(G) \leq \alpha n$ contains a C_k-factor.

- This implies a result of Böttcher, Parczyk, Sgueglia and Skokan on cycle-factors in a *random perturbation* model (arXiv:2103.06136).

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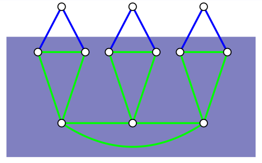
The absorption method

The absorption method was firstly introduced by Rödl, Ruciński and Szemerédi and it is an important tool for studying the existence of spanning structures in graphs, digraphs and hypergraphs. A crucial ingredient is to build an absorbing set.

Widely used constructions of absorbing sets by Rödl, Ruciński and Szemerédi, or independently by Hàn, Person, and Schacht rely on the property that *every k-subset in $V(G)$ has polynomially many absorbers of a certain type.*



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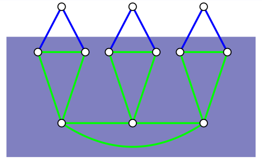

Lemma (Nenadov–Pehova, '18)

Given a constant $\gamma > 0$, $k, t \in \mathbb{N}$ and a k-vertex graph H, there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that if G is an n-vertex graph with $n \geq n_0$ such that for every $S \in \binom{V(G)}{k}$ there is a family of at least γn vertex-disjoint (H, t)-absorbers, then G contains a ξ-absorbing set of size at most γn.

- **The bipartite template** (introduced by Montgomery) guarantees the existence of an absorbing set, *provided that every k-set in $V(G)$ has linearly many vertex-disjoint absorbers.*
- **The lattice-based absorbing method** (developed by Han and Keevash et al.) is essentially used to detect which kinds of k-sets have linearly many vertex-disjoint absorbers.

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Thanks for listening!