Density of C_{-4}-critical signed graphs

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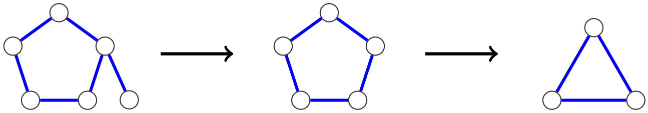
H-coloring

- A homomorphism of a graph G to a graph H is a mapping from $V(G)$ to $V(H)$ such that the adjacency is preserved.
- If G admits a homomorphism to H, then we say G admits an H-coloring or G is H-colorable.



4-color theorem restated

Every planar graph admits a K_4-coloring.

$(2k+1)$-coloring problem vs C_{2k+1}-coloring problem

$T_k(G)$: the graph obtained from G by replacing each edge uv with a path of length k.

Indicator construction Lemma [P. Hell and J. Nešetřil 1990]

A graph G is $(2k+1)$-colorable if and only if $T_{2k-1}(G)$ is C_{2k+1}-colorable.

- The C_{2k+1}-coloring problem is NP-complete. [H.A. Maurer, J.H. Sudborough, E. Welzl 1981]

Can we make use of even cycles to capture $2k$-coloring problem?

Signed graphs

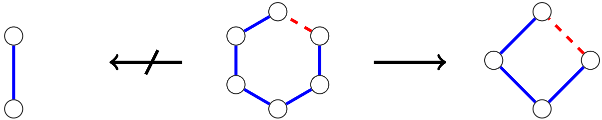
- A signed graph is a graph $G = (V, E)$ together with an assignment $\sigma : E(G) \to \{+, -\}$, denoted by (G, σ).
- A switching at a vertex v is to switch the signs of all the edges incident to this vertex.
- We say (G, σ') is switching equivalent to (G, σ) if it is obtained from (G, σ) by switching at some vertices (allowing repetition).
- The sign of a closed walk is the product of signs of all the edges of this walk.

Theorem [T. Zaslavsky 1982]

Signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles.

Homomorphism of signed graphs

- A homomorphism of a signed graph (G, σ) to (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$, respectively, such that the adjacency, the incidence and the signs of closed walks are preserved. If there is one, we write $(G, \sigma) \to (H, \pi)$.
- A homomorphism is edge-sign preserving if it, furthermore, preserves the signs of the edges. If there is one, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.



Proposition [R. Naserasr, É. Sopena, and T. Zaslavsky 2021]

$(G, \sigma) \to (H, \pi) \Leftrightarrow \exists\, \sigma' \equiv \sigma,\ (G, \sigma') \xrightarrow{s.p.} (H, \pi)$.

No-homomorphism Lemma

There are four possible types of closed walks in signed graphs:

- type 00 is a closed walk which is positive and of even length,
- type 01 is a closed walk which is positive and of odd length,
- type 10 is a closed walk which is negative and of even length,
- type 11 is a closed walk which is negative and of odd length.

The length of a shortest nontrivial closed walk in (G, σ) of type ij, $(ij \in \mathbb{Z}_2^2)$, is denoted by $g_{ij}(G, \sigma)$.

No-homomorphism Lemma [R. Naserasr, É. Sopena, and T. Zaslavsky 2021]

If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$.

k-coloring problem vs C_{-k}-coloring problem

$T_k(G,\sigma)$: a signed graph obtained from (G,σ) by replacing each edge uv with a signed path of length k with sign $-\sigma(uv)$.

Lemma [R. Naserasr, L-A. Pham, and Z. Wang 2022]

A graph G is k-colorable if and only if $T_{k-2}(G,+)$ is C_{-k}-colorable.

In particular, the $2k$-coloring problem of graphs is captured by the C_{-2k}-coloring problem of signed bipartite graphs.

Special case when $k = 4$

A graph G is 4-colorable if and only if $T_2(G,+)$ is C_{-4}-colorable.

Proof of $G \to K_4 \Leftrightarrow T_2(G,+) \to C_{-4}$

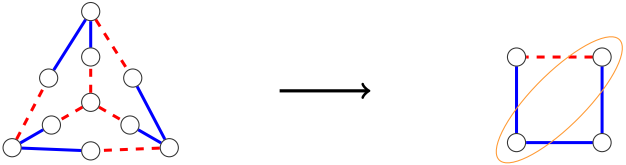


Figure: $G \to K_4 \Rightarrow T_2(G,+) \to C_{-4}$

- $\Rightarrow$: It suffices to show that $T_2(K_4,+) \to C_{-4}$.
- $\Leftarrow$: Let $\varphi : T_2(G,+) \to C_{-4}$. This mapping preserves the bipartition.

Edge-sign preserving homomorphism to C_{-4}

Lemma [C. Charpentier, R. Naserasr, and E. Sopena 2020]

Given a signed bipartite graph (G, σ),

$$(G, \sigma) \xrightarrow{s.p.} C_{-4} \Leftrightarrow (P_3, \pi) \not\subseteq (G, \sigma).$$

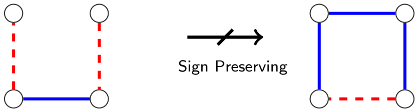


Figure: C_{-4} and its edge-sign preserving dual

NP-completeness of C_{-4}-coloring problem

- In order to map a signed bipartite graph (G, σ) to C_{-4}, it is necessary and sufficient to find an equivalent signature σ' of σ where no positive edge is incident with a negative edge at each of its end.
- Deciding whether there exists an edge-sign preserving homomorphism to C_{-4} is in polynomial time but finding such an equivalent signature is hard.
- The C_{-4}-coloring problem is NP-complete. [R. C. Brewster, F. Foucaud, P. Hell and R. Naserasr 2017]

k-critical and H-critical

- A graph is k-critical if it is k-chromatic but any proper graph of it is $(k-1)$-colorable.
- A graph is H-critical if it is not H-colorable but any proper graph of it is H-colorable. [P. A. Catlin 1988]

One of the most popular questions of H-critical graphs on n vertices is to find the lower bound for the number of edges as a function of n.

- Any C_3-critical graph on n vertices has at least $\frac{5n-2}{3}$ edges; [A. Kostochka and M. Yancey 2014]
- Any C_5-critical graph on n vertices has at least $\frac{5n-2}{4}$ edges; [Z. Dvorak and L. Postle 2017]
- Any C_7-critical graph on n vertices has at least $\frac{17n-2}{15}$ edges. [L. Postle and E. Smith-Roberge 2022]

(H, π)-critical signed graph

Definition [R. Naserasr, L-A. Pham, and Z. Wang 2022]

A signed graph (G, σ) is (H, π)-critical if the following hold:

- $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$, for $ij \in \mathbb{Z}_2^2$;
- $(G, \sigma) \not\to (H, \pi)$;
- $(G', \sigma) \to (H, \pi)$ for any proper subgraph $(G', \sigma) \subset (G, \sigma)$.

We observe that:

- A graph G is k-critical $\Leftrightarrow$ $(G, +)$ is $(K_{k-1}, +)$-critical.
- By No-homomorphism Lemma, the first condition eliminates trivial cases.

C_{-4}-critical signed graph

A signed graph (G, σ) is C_{-4}-critical if the following hold:

- (G, σ) is bipartite and of negative-girth at least 4;
- $(G, \sigma) \not\to C_{-4}$;
- $(G', \sigma) \to C_{-4}$ for any proper subgraph $(G', \sigma) \subset (G, \sigma)$.

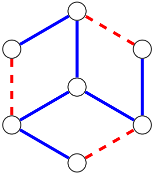


Figure: $\hat{W}$

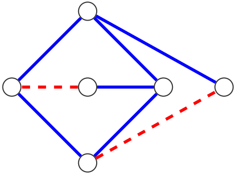


Figure: Γ

Jaeger-Zhang Conjecture

Jaeger-Zhang Conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth at least $4k+1$ admits a homomorphism to C_{2k+1}.

- $k=1$: Grötzsch's theorem;
- $k=2$: verified for odd-girth 11 [Z. Dvořák and L. Postle 2017; D. Cranston and J. Li 2020];
- $k=3$: verified for odd-girth 17 [D. Cranston and J. Li 2020; L. Postle and E. Smith-Roberge 2022];
- $k \geq 4$:
  - verified for odd-girth $8k-3$ [X. Zhu 2001];
  - verified for odd-girth $\frac{20k-2}{3}$ [O.V. Borodin, S.-J. Kim, A.V. Kostochka and D.B. West 2002];
  - verified for odd-girth $6k+1$ [L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013].

Signed bipartite analog of Jaeger-Zhang Conjecture

Signed bipartite analog of Jaeger-Zhang Conjecture [R. Naserasr, E. Rollová, É. Sopena 2015]

Every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to C_{-2k}.

- $k = 2$: 8 is the best negative-girth condition [R. Naserasr, L-A. Pham, and Z. Wang 2022];
- $k = 3, 4$: verified for negative-girth 14 and 20 [J. Li, Y. Shi, Z. Wang and C. Wei 2022+];
- $k \geq 5$:
  - verified for negative-girth $8k - 2$ [C. Charpentier, R. Naserasr, and E. Sopena 2020];
  - verified for negative-girth $6k - 2$ [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+].

Density of C_{-4}-critical signed graphs

Theorem [R. Naserasr, L-A. Pham, and Z. Wang 2022]

If $\hat{G}$ is a C_{-4}-critical signed graph which is not isomorphic to $\hat{W}$, then

$$e(\hat{G}) \geq \frac{4v(\hat{G})}{3}.$$

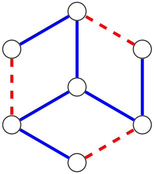


Figure: $\hat{W}$

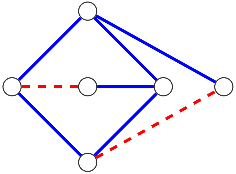


Figure: Γ

Potential method

The potential of a signed graph $\hat{G}$ is defined to be

$$p(\hat{G}) = 4v(\hat{G}) - 3e(\hat{G}).$$

Theorem [R. Naserasr, L-A. Pham, and Z. Wang 2022]

If $\hat{G}$ is C_{-4}-critical and $\hat{G} \neq \hat{W}$, then $p(\hat{G}) \leq 0$.

We will estimate the potentials of some subgraphs of the minimum counterexample and list some forbidden configurations in it.

Potential method

Let $\hat{G} = (G, \sigma)$ be the minimum counterexample with respect to $v(\hat{G}) + e(\hat{G})$.

- $\hat{G}$ is a C_{-4}-critical signed graph which is not isomorphic to $\hat{W}$ and it satisfies $p(\hat{G}) \geq 1$;
- For any C_{-4}-critical signed graph $\hat{H}$ with $\hat{H} \neq \hat{W}$ satisfying that $v(\hat{H}) < v(\hat{G})$, $p(\hat{H}) \leq 0$.

Observations:

- $\hat{G}$ is 2-connected.
- There must exist a 2-vertex in $\hat{G}$.
- There is no 3-thread in $\hat{G}$.

Key Lemma

$P_2(\hat{H})$: a graph obtained from $\hat{H}$ by adding a vertex v and joining it with two vertices in $\hat{H}$ (with any signature).

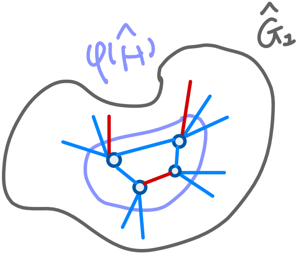
Lemma (Potential of subgraphs)

Let $\hat{G} = (G, \sigma)$ be a minimum counterexample and let $\hat{H}$ be a subgraph of $\hat{G}$. Then

1. $p(\hat{H}) \geq 1$ if $\hat{G} = \hat{H}$;
2. $p(\hat{H}) \geq 3$ if $\hat{G} = P_2(\hat{H})$;
3. $p(\hat{H}) \geq 4$ otherwise.

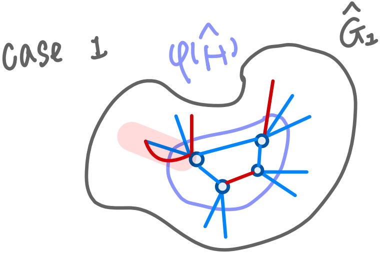
Sketch of the proof

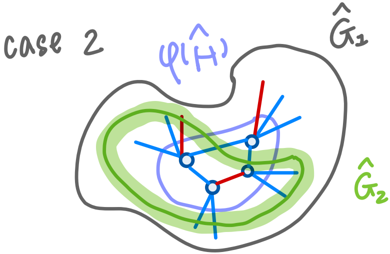
- Suppose to the contrary that $\hat{G}$ contains a proper subgraph $\hat{H}$ which does not satisfy $\hat{G} = P_2(\hat{H})$, and satisfies $p(\hat{H}) \leq 3$. We take the maximum such $\hat{H}$.
- Note that $\hat{H}$ is a proper induced subgraph of order at least 5. Let φ be a mapping of $\hat{H}$ to C_{-4}.



Sketch of the proof

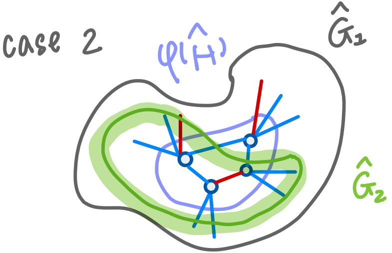
- Define $\hat{G}_1$ to be a signed (multi)graph obtained from $\hat{G}$ by first identifying vertices of $\hat{H}$ which are mapped to a same vertex of C_{-4} under φ and identifying the parallel edges of the same sign. We conclude that $\hat{G}_1 \not\to C_{-4}$.
- Two possibilities: Either $\hat{G}_1$ contains a C_{-2}, or $\hat{G}_1$ contains a C_{-4}-critical subgraph $\hat{G}_2$.

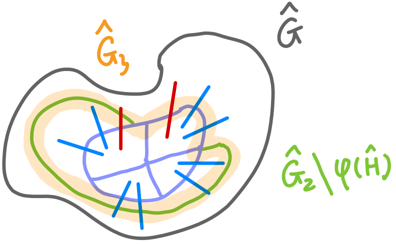




Sketch of the proof

- Case 1: $\hat{G}_1$ contains a C_{-2}. Then by computing $p(\hat{H} + P_{-2})$ and using the maximality of $\hat{H}$, we obtain the contradiction.
- Case 2: $\hat{G}_1$ contains a C_{-4}-critical subgraph $\hat{G}_2$. First of all, by the minimality of $\hat{G}$, we have $p(\hat{G}_2) \leq 0$. Then we define a signed graph $\hat{G}_3$ (subgraph of $\hat{G}$) by combing $\hat{G}_2$ and $\hat{H}$ with some modifications. By the relation of $\hat{H} \subsetneq \hat{G}_3 \subset \hat{G}$, it leads to a contradiction.





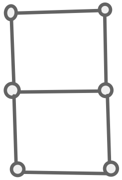
Forbidden configurations

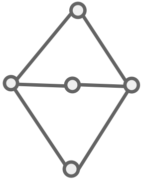
Lemma

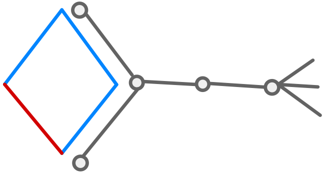
Two 4-cycles in the minimum counterexample $\hat{G}$ cannot share one edge or two edges.

Lemma

Let vv_1u be a 2-thread in the minimum counterexample $\hat{G}$. Suppose that v is a 3-vertex and let v_2, v_3 be the other two neighbors of v. Then the path v_2vv_3 must be contained in a negative 4-cycle in $\hat{G}$.



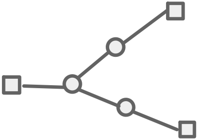




Forbidden configurations and discharging technique

Lemma

A vertex of degree 3 in the minimum counterexample $\hat{G}$ does not have two neighbors of degree 2.



Forbidden

Constructions of C_{-4}-critical signed graphs of density $\frac{4}{3}$

$\tilde{G}$: a signed graph obtained by replacing each edge of G by C_{-2}.

Lemma [R. Naserasr, L-A. Pham, and Z. Wang 2022]

A graph G is $(k+1)$-critical if and only if $T_{2k-2}(\tilde{G})$ is C_{-2k}-critical.

As odd cycles are the only 3-critical graphs, $T_2(\tilde{C}_{2k+1})$, for each $k \geq 1$, is a C_{-4}-critical signed graph whose density is $\frac{4}{3} = \frac{8k+4}{6k+3}$.

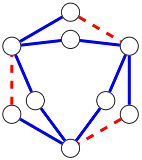


Figure: $T_2(\tilde{C}_3)$

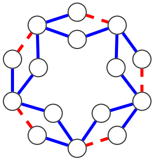


Figure: $T_2(\tilde{C}_5)$

Constructions of sparse C_{-4}-critical signed graphs

Let $\hat{G}_1$ and $\hat{G}_2$ be two C_{-4}-critical signed graphs.

- Assuming that there is a 2-vertex u in $\hat{G}_1$ with u_1, u_2 being its neighbors and a 2-vertex v in $\hat{G}_2$ with v_1, v_2 being its neighbors, we build a signed graph $\mathcal{F}(\hat{G}_1, \hat{G}_2)$ from disjoint union of $\hat{G}_1$ and $\hat{G}_2$ by deleting u and v, and adding a positive edge u_1v_1 and a negative edge u_2v_2.

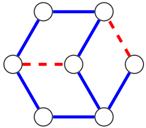


Figure: $\hat{W}$

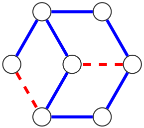


Figure: $\hat{W}$

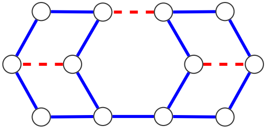


Figure: $\mathcal{F}(\hat{W}, \hat{W})$

Constructions of sparse C_{-4}-critical signed graphs

Analog of Hajós construction

- Assuming that there is a positive edge x_1y_1 in $\hat{G}_1$ and a negative edge x_2y_2 in $\hat{G}_2$, we build a signed graph $\mathcal{H}(\hat{G}_1, \hat{G}_2)$ from disjoint union of $\hat{G}_1$ and $\hat{G}_2$ by deleting x_1y_1, x_2y_2 and identifying x_1 with x_2 and y_1 with y_2.

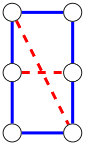


Figure: Γ

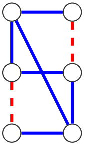


Figure: Γ

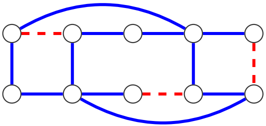


Figure: $\mathcal{H}(\Gamma, \Gamma)$

Mapping signed bipartite planar graphs to C_{-4}

A signed graph (G, σ) is 2k-colorable if there exists a mapping $c : V(G) \to \{\pm 1, ..., \pm k\}$ such that for each edge uv of (G, σ), $c(u) \neq \sigma(uv)c(v)$.

It has been conjectured in [E. Máčajová, A. Raspaud, M. Škoviera 2016] that every signed planar simple graph is 4-colorable.

Theorem [F. Kardoš and J. Narboni 2021]

There exists a signed planar simple graph which is not 4-colorable.

Mapping signed bipartite planar graphs to C_{-4}

Theorem [F. Kardoš and J. Narboni 2021]

There exists a signed planar simple graph which is not 4-colorable.

Lemma [R. Naserasr, L-A. Pham, and Z. Wang 2022]

A signed graph (G, σ) is $2k$-colorable if and only if $T_{2k-2}(G, \sigma)$ is C_{-2k}-colorable.

When $k = 2$, there exists a signed graph $T_2(G, \sigma)$ which does not admit a homomorphism to C_{-4}.

Theorem [R. Naserasr, L-A. Pham, and Z. Wang 2022]

There exists a bipartite planar graph G of girth 6 with a signature σ such that $(G, \sigma) \nrightarrow C_{-4}$.

Mapping signed bipartite planar graphs to C_{-4}

- By Folding Lemma, starting from a signed bipartite planar graph whose shortest negative cycles are of length at least 8, we get a homomorphic image $\hat{G}$ with a planar embedding where all faces are negative 8-cycles.
- Applying Euler's Formula on this graph, we have $|E(G)| \leq \frac{3(|V(G)|-2)}{4}$.

Theorem [R. Naserasr, L-A. Pham, and Z. Wang 2022]

Every signed bipartite planar graph of negative-girth at least 8 admits a homomorphism to C_{-4}. Moreover, the girth condition is the best possible.

1. Introduction
   - H-coloring of graphs
   - Homomorphism of signed graphs
   - (H, π)-critical signed graphs
   - Jaeger-Zhang conjecture and its bipartite analog
2. Density of C_{-4}-critical signed graphs
   - C_{-4}-critical signed graphs
   - Application to the planarity
3. Conclusion

Relation with circular coloring of signed graphs

- Recently we have defined the notion of circular chromatic numbers of signed graphs. We prove that for any signed bipartite graph (G, σ),

$$X_c(G, \sigma) \leq \frac{8}{3} \Leftrightarrow (G, \sigma) \to C_{-4}.$$

- So our work can be restated as: Any $\frac{8}{3}$-critical signed bipartite graph on n vertices has at least $\frac{4n}{3}$ edges except for $\hat{W}$.

Discussion

- We look for some strong sufficient conditions for signed bipartite planar graphs mapping to C_{-4}.

Conjecture

Let G be a bipartite planar graph of girth at least 6. Let σ be a signature on G such that in (G, σ) all 6-cycles are of a same sign. Then $(G, \sigma) \to C_{-4}$.

- If it is true, it implies the 4-color theorem by T_2-construction on a planar simple graph.

Discussion

- We determined that the best girth condition for mapping signed bipartite planar graphs to C_{-4} is 8 rather than 6.

Question

What is the best negative-girth condition for signed bipartite planar graphs mapping to C_{-2k}?

The end. Thank you!