INDEPENDENT DOMINATION OF GRAPHS WITH MAXIMUM DEGREE

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& KAIST

with
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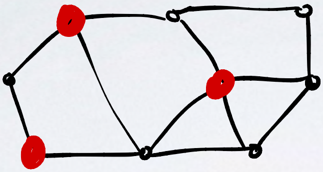
https://dimag.ibs.re.kr/home/sangil/

INDEPENDENT SETS IN A GRAPH

- $X \subseteq V(G)$ is independent (stable) if no 2 vertices in X are adjacent.



● independent

MAXIMUM INDEPENDENT SET

$|X|$ is max among independent sets X

MAXIMAL INDEPENDENT SET

X is independent but if $Y \supsetneq X$, then Y is not indep

INDEPENDENT SETS vs MAXIMUM DEGREE

$\Delta(G)$ = maximum degree of vertices in G

MAXIMUM INDEPENDENT SET

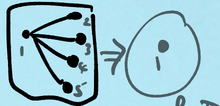
$\alpha(G)$ = size of a maximum independent set

If $\Delta(G) \le \Delta$, then

$$\alpha(G) \ge \frac{n}{\Delta+1}$$

$$\left(\Delta(G) \le \Delta \Rightarrow \chi(G) \le \Delta+1 \Rightarrow \alpha(G) \ge \frac{n}{\Delta+1} \right)$$

WHAT IS A GOOD QUESTION FOR MAXIMAL INDEPENDENT SETS?



MAX SIZE OF A MAXIMAL INDEP SETS = SIZE OF A MAXIMUM INDEP SET $= \alpha(G)$

How about

MIN SIZE OF A MAXIMAL INDEP SET?

$i(G)$

MAXIMAL INDEPENDENT ⟺ INDEPENDENT DOMINATING

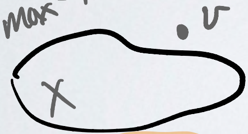
OLD THM (IN BOOKS ... BERGE 1962, ORE 1962)

MAXIMAL INDEPENDENT ⟺ INDEPENDENT DOMINATING

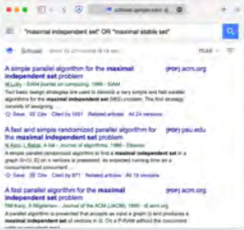
$\forall v \in V(G)$, $v \in X$ or $N(v) \cap X \neq \emptyset$

PROOF

max & indep

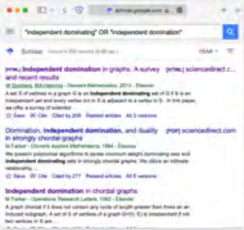


THUS, EVERY GRAPH HAS AN INDEPENDENT DOMINATING SET.



MAXIMAL INDEPENDENT

~ 14200 papers



INDEPENDENT DOMINATING

~ 4500 papers

SURPRISINGLY MANY BOOKS & PAPERS ON DOMINATING SETS

books (Berge (1962)
Ore (1962)

X is a dominating set if $\forall v \in V(G)$, $v \in X$ or v is adjacent to a vertex in X.

Towards a Theory of Domination in Graphs

E. J. Cockayne
University of Victoria
Victoria, B.C., Canada

S. T. Hedetniemi
University of Oregon
Eugene, Oregon

ABSTRACT

This paper presents a quick review of results and applications concerning dominating sets in graphs. The domatic number of a graph is defined and studied. It is seen that the theory of domination resembles the well known theory of colorings of graphs.

1. INTRODUCTION

A set of vertices D in a graph G = (V,E) is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. A set of vertices D is *independent* if no two vertices in D are adjacent.

The concepts of domination and independence have existed for a long time, two earlier discussions may be found in Ore [19, Ch.13] and Berge [2]. A more up-to-date discussion, and perhaps the most comprehensive, on the subjects of independence and domination can be found in Berge [3]. We note that Berge has used the terms *stable* or *interiorly stable* for what we call independent and has used *absorbant* or *externally stable* for our term dominating.

The literature contains many papers dealing with the theory of independent sets and the related topic of graph colouring. However, there are few papers about domination in graphs, which fact is somewhat surprising particularly in view of the diversity of applications. A brief survey of the literature reveals the following sample of applications for the concept of a dominating set.

In [19, p.207], Ore mentions the problem of placing a minimum number of queens on a chessboard so that each square is controlled by at least one queen.

Networks, 7: 247-261
 247

1977

Discrete Mathematics 86 (1990) 257-277
North-Holland 257

BIBLIOGRAPHY ON DOMINATION IN GRAPHS AND SOME BASIC DEFINITIONS OF DOMINATION PARAMETERS

S.T. HEDETNIEMI and R.C. LASKAR
Department of Computer Science and Mathematical Sciences, Clemson University, Clemson, SC 29631, USA

Received 2 December 1988

Introduction

The following bibliography on Domination in Graphs has been compiled over the past six years at Clemson University, where we regularly maintain a computer data base on this topic. Several people have been especially helpful in keeping this bibliography up-to-date and we would like to thank them: E.J. Cockayne, Victoria, British Columbia; P.J. Slater, Huntsville, Alabama; Maciej Syslo, Wroclaw, Poland; Bohdan Zelinka, Liberec, Czechoslovakia; E. Sampathkumar, Dharwad, India; A. Brandstädt, Rostock, GDR; and Peter Hammer, New Brunswick, New Jersey.

This bibliography essentially starts with the graph theory texts of König (1950), Berge (1958) and Ore (1962). Although a few research papers on domination were published between 1958 and 1975, a survey paper by Cockayne and Hedetniemi (1975) served to focus attention on the subject sufficiently to 'get the ball rolling'. By 1988 the domination bibliography included well over 300 citations, about one-third of which are concerned with algorithms for computing various domination numbers for special classes of graphs.

In our view, the rapid growth in the number of domination papers is attributable largely to three factors:

(i) the diversity of applications to both real-world and other mathematical 'covering' or 'location' problems;

(ii) the wide variety of domination parameters that can be defined;

(iii) the NP-completeness of the basic domination problem, its close and 'natural' relationships to other NP-complete problems, and the subsequent interest in finding polynomial time solutions to domination problems in special classes of graphs.

Thus we expect that this bibliography will continue to grow at a steady rate.

As far as we know, only four survey papers have been written on domination in graphs:

Cockayne and Hedetniemi, 1975;

0012-365X/90/$03.50 © 1990 — Elsevier Science Publishers B.V. (North-Holland)

1990

Discrete Mathematics 313 (2013) 839–854

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journal homepage: www.elsevier.com/locate/disc

Perspective

Independent domination in graphs: A survey and recent results

Wayne Goddard [a,*], Michael A. Henning [b]

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[b] Department of Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa

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ABSTRACT

A set S of vertices in a graph G is an independent dominating set of G if S is an independent set and every vertex not in S is adjacent to a vertex in S. In this paper, we offer a survey of selected recent results on independent domination in graphs.

1. Introduction

An independent dominating set in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. Independent dominating sets have been studied extensively in the literature. In this paper, we survey selected results on independent domination in graphs.

Dominating and independent dominating sets. A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum size of a dominating set.

A set is *independent* (or *stable*) if no two vertices in it are adjacent. An *independent dominating set* of G is a set that is both dominating and independent in G. The *independent domination number* of G, denoted by $i(G)$, is the minimum size of an independent dominating set. The *independence number* of G, denoted $\alpha(G)$, is the maximum size of an independent set in G. It follows immediately that $\gamma(G) \le i(G) \le \alpha(G)$.

A dominating set of G of size $\gamma(G)$ is called a γ-set, while an independent dominating set of G of size $i(G)$ is called an i-set.

History. The idea of an independent dominating set arose in chessboard problems. In 1862, de Jaenisch [30] posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. A graph G may be formed from an 8×8 chessboard by taking the squares as the vertices with two vertices adjacent if a queen situated on one square attacks the other square. The graph G is known as the *queens graph*. The minimum number of mutually non-attacking queens that attack all the squares of a chessboard is the independent domination number $i(G)$. For the queens graph G, we note that $\alpha(G) = 8$, $i(G) = 7$, and $\gamma(G) = 5$.

The theory of independent domination was formalized by Berge [6] and Ore [91] in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [21,22].

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ABSTRACT

A set S of vertices in a ... set and every vertex not in S is adjacent to a vertex ... selected recent results on independent domination in graphs.

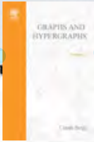
2. Bounds on the independent domination

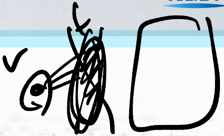
2.1. General bounds

The first result establishes a simple relationship between the independent domination number and the maximum degree of a graph, and was given by Berge [7]. ?

Proposition 2.1 ([7]). *For a graph G with n vertices and maximum degree Δ,*

$$\left\lceil \frac{n}{1+\Delta} \right\rceil \le i(G) \le n - \Delta.$$





1. Introduction

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The theory of independent domination was formalized by Berge [6] and Ore [91] in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [21,22].

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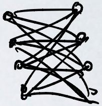
- $i(K_{\Delta+1}) = 1$
- $i(K_{\Delta,\Delta}) = \Delta$



Domke, Dunbar, Markus (1997)

Let $n = |V(G)|$.

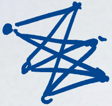
- $i(G) \le n - \Delta$.
- Characterize "=" condition for bipartite graphs

$$\begin{cases} \Delta = \max(|A|, |B|) \\ i = \min(|A|, |B|) \end{cases}$$

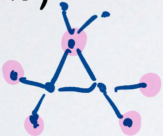
CAN WE IMPROVE "$i(G) \le n - \Delta$" ?

THM [AKBARI, AKBARI, DOOSTHOSSEINI, HADIZADEH, HENNING, NARAGHI (2022)]

$i(G) \le \frac{1}{2} n$ if $\Delta(G) \le 3$, no isolated vertex



THM [CHO, CHOI, PARK 2021+]



$i(G) \le \frac{5}{9} n$ if $\Delta(G) \le 4$ no isolated vertex

CONJECTURE (CHO, CHOI, PARK 2021+)

$$i(G) \le \left(1 - \frac{\Delta}{\left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor}\right) n \quad \text{if } \Delta(G) \le \Delta$$

no isolated vertex

Best possible if true: (?)

$i(G) = \lceil \frac{\Delta}{2} \rceil \lceil \frac{\Delta}{2} \rceil + 1$

$n = \left(\frac{\Delta}{2}+1\right)\left(\frac{\Delta}{2}+1\right)$

CONJECTURE (CHO, CHOI, PARK 2021+)

$$i(G) \le \left(1 - \frac{\Delta}{\left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor}\right) n \quad \text{if} \quad \Delta(G) \le \Delta$$

no isolated vertex

$i(G) = \lceil \frac{\Delta}{2} \rceil \lceil \frac{\Delta}{2} \rceil + 1$

$n = (\frac{\Delta}{2}+1)(\frac{\Delta}{2}+1)$

PREVIOUSLY

PROVED FOR $\Delta \le 8$ (Cho, Choi, Park)

WRITTEN FOR $\Delta = 4$

Our Theorem

1. Prove this conjecture.
   Characterize the equality condition when $\Delta \ge 4$

2. Prove a stronger theorem
   having an arbitrary large connected tight graph

$$\frac{\Delta}{\left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor} = \frac{\Delta}{\left\lfloor \frac{\Delta^2}{4} \right\rfloor + \Delta + 1}$$

OUR THM [Cho, Kim, Kim, O.]

- Let $\Delta \geq 4$.

  G: connected graph, maximum degree $\leq \Delta$.

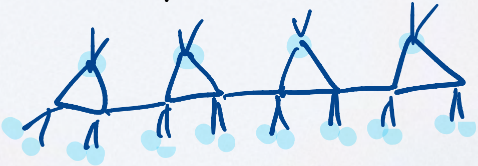
  (i) If G is Δ-special, then

  $$i(G) = \left(1 - \frac{\Delta}{\left\lfloor \frac{\Delta^2}{4} \right\rfloor + \Delta}\right) n + \frac{\Delta}{\left\lfloor \frac{\Delta^2}{4} \right\rfloor + \Delta}$$

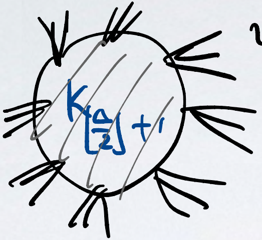
  (ii) If G is not Δ-special, then

  $$i(G) \leq \begin{cases} \left(1 - \frac{\Delta}{\left\lfloor \frac{\Delta^2}{4} \right\rfloor + \Delta}\right) n & \text{if } \Delta \neq 5, \\ \frac{5}{9} n & \text{if } \Delta = 5. \end{cases}$$

$$i(G) \leq \left(1 - \frac{\Delta}{\left\lfloor \frac{\Delta^2}{4} \right\rfloor + \Delta}\right)(n-1) + 1$$



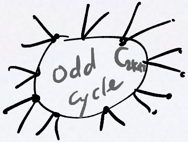
What are Δ-special graphs?

- K_1 $i=1,\ n=1$
- 

$i = 1 + \lfloor \frac{\Delta}{2} \rfloor \lceil \frac{\Delta}{2} \rceil$

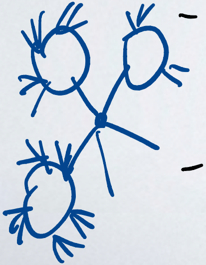
$n = (\lfloor \frac{\Delta}{2} \rfloor + 1)(\lceil \frac{\Delta}{2} \rceil + 1)$

or

- If $\Delta = 4$, 

$i = (k-1) + 2(k+1)$

$n = 3(2k+1)$



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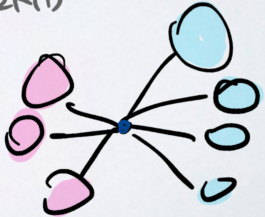
$\Longrightarrow$



Joining 2 Δ-special graphs

OUR THM [Cho, Kim, Kim, O]

- Let $\Delta \ge 4$.

G: connected graph, maximum degree $\le \Delta$.

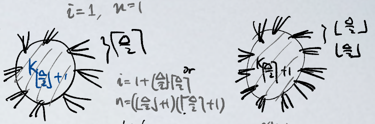
(i) If G is Δ-special, then

$$i(G) = \left(1-\frac{\Delta}{\lfloor \frac{\Delta^2}{4}\rfloor+\Delta}\right)n + \frac{\Delta}{\lfloor \frac{\Delta^2}{4}\rfloor+\Delta}$$

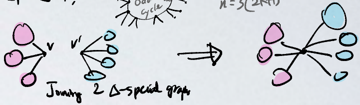
(ii) If G is not Δ-special, then

$$i(G) \le \begin{cases} \left(1-\frac{\Delta}{\lfloor \frac{\Delta^2}{4}\rfloor+\Delta}\right)n & \text{if } \Delta \ne 5, \\ \frac{5}{9}n & \text{if } \Delta = 5. \end{cases}$$

What are Δ-special graphs?

- K_1 $\quad i=1,\ n=1$
- 

- If $\Delta=4$, 

$i = (k-1)+2(k+1)$
$n = 3(2k+1)$
- 

COR1: $\Delta \ge 4$, $\Delta \ne 5$, $\Delta(G) \le \Delta$

$$\Rightarrow\ i(G) \le \left(1-\frac{\Delta}{\lfloor\frac{\Delta^2}{4}\rfloor+\Delta}\right)(n-1)+1$$

Why $\Delta=5$ so strange?

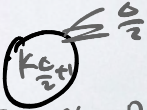
$$\frac{i}{n} = \frac{5}{9} > \left(1-\frac{5}{\lfloor\frac{5^2}{4}\rfloor+5}\right) = \frac{6}{11}$$

COR2: $\Delta \ge 4$, G: connected

- $\Delta(G) \le \Delta$
- not isolated vertex

$$\Rightarrow\ i(G) \le \left(1-\frac{\Delta}{\lfloor\frac{(\Delta+2)^2}{4}\rfloor}\right)n$$

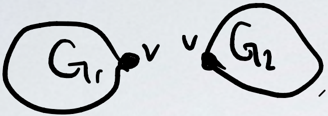
The equality holds

$\Leftrightarrow$ $G =$ 

Proving the Conj of Cho, Choi, Park.

KEY LEMMAS

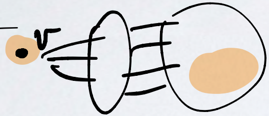
Lemma 1:



$\exists S_1$, minimum indep. dom set of G_1 containing v

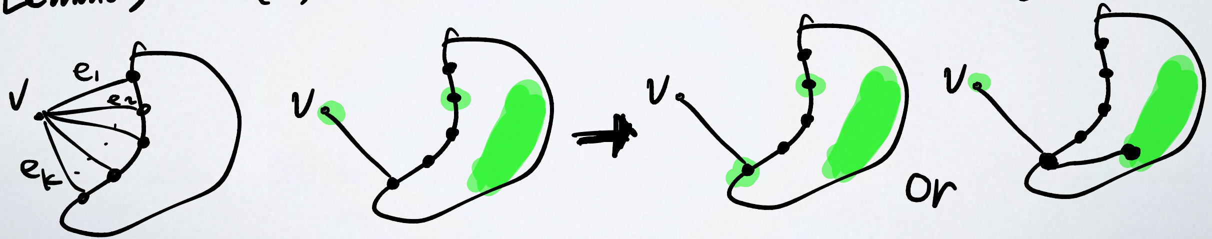
$\Longrightarrow i(G_1 \cdot G_2) = i(G_1) + i(G_2) - 1$

Lemma 2:



$i(G) \leq i(G - v - N(v)) + 1$

Lemma 3: $i(G) \leq i(G - e_1 - e_2 - \cdots - e_{k-1})$ if $k = \deg v \geq 2$

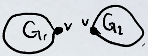


Proof Sketch

Lemma1 $\Rightarrow$ Δ-special graphs. ✓

WMA G is not Δ-special.

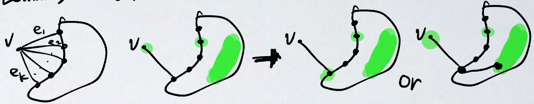
$n_\Delta(G)$ = # Δ-special components

Lemma1:  $\exists S_1$, minimum indep. dom set of G_1 containing v

$\Rightarrow i(G_1 \cdot G_2) = i(G_1) + i(G_2) - 1$

Lemma 2:  $i(G) \le i(G - v - N(v)) + 1$

Lemma3: $i(G) \le i(G - e_1 - e_2 - \cdots - e_{k-1})$ if $k = \deg v \ge 2$



CLAIM: $$i(G) \le (1-t)\,n + t\, n_\Delta(G)$$

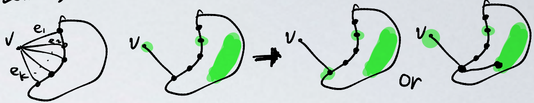
$$t = \begin{cases} \dfrac{\Delta}{\lfloor \Delta^2/4 \rfloor + \Delta} & \text{if } \Delta \ne 5, \\ \dfrac{4}{9} & \text{if } \Delta = 5. \end{cases}$$

WMA $n_\Delta(G)=0$ (G is not Δ-special)

G: conn.

Lemma 2:  $i(G) \le i(G-v-N(v))+1$

Lemma 3: $i(G) \le i(G-e_1-e_2-\cdots-e_{k-1})$ if $k=\deg v \ge 2$



Lemma: If every nbr of v has no vertex of deg 1, and degree of $v \ge 2$ then $i(G) \le (1-t)n$

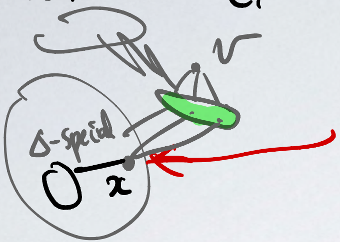
By lemma 3, $i(G) \le i(H_j)$ $H_j := G$ − all edges incident with v except e_j

$\Rightarrow$ WMA H_j has a Δ-special component for all j

(otherwise $i(G) \le i(H_j) \le (1-t)n$)

$\Rightarrow$ G is Δ-special!

Case 1: $G-v-N(v)$ has a Δ-special component C of size >1



x: vertex in C, distance 2 from v

$\Rightarrow$ Every nbr of x in C has degree Δ in C (by the def. of Δ-special graphs)

$\Rightarrow$ No nbr of x has degree $=1$ in G

$\Rightarrow$ By Lemma, $i(G) \leq (1-t)n$.

Case 2: $G-v-N(v)$ has no non-trivial Δ-special components for every vertex v.

Let $n_0(H) = \#$ isolated vertices in H.

Lemma 2 $\Longrightarrow$

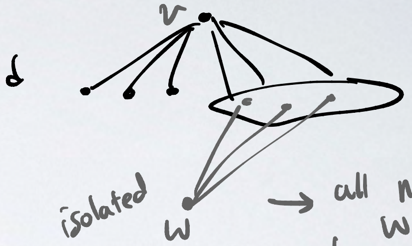
$$i(G) \le i(G-v-N(v)) + 1$$

$$\le (1-t)(n-d(v)-1) + t\, n_0(G-v-N(v)) + 1$$

$$\overset{?}{\le} (1-t)\, n$$

WMA $n_0(G-v-N(v))$ big

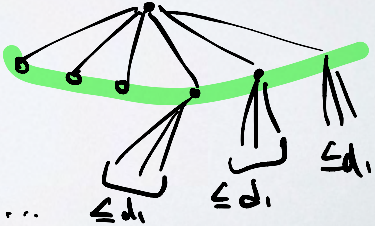
Choose v so that
$d_1 = \#$ degree-1 nbrs of v
is maximized.



$\rightarrow$ all nbrs of w has $\deg \ge 2$

$$n_0(G-v-N(v)) \le (d(v)-d_1)\, d_1$$

$$\le \left(\frac{d(v)}{2}\right)^2 \cdots$$



OPEN PROBLEMS

What if
G has minimum degree $\geq \delta$
and
maximum degree $\leq \Delta$?

THANK YOU FOR YOUR ATTENTION!