Some results on the connectedness of friends-and-strangers graphs

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The Definition

Definition (Defant-Kravitz, Comb. Theory 1 (2021))

Let X and Y be two graphs, each with n vertices. The friends-and-strangers graph $\mathsf{FS}(X, Y)$ *of X and Y is a graph with vertex set consisting of all bijections from $V(X)$ to $V(Y)$, two such bijections σ, σ' are adjacent iff they differ precisely on two adjacent vertices, say $a, b \in X$ with $\{a, b\} \in E(X)$, and the corresponding mappings are adjacent in Y, i.e.,*

- $\{\sigma(a), \sigma(b)\} \in E(Y)$;
- $\sigma(a) = \sigma'(b)$, $\sigma(b) = \sigma'(a)$ *and* $\sigma(c) = \sigma'(c)$ *for all* $c \in V(X) \backslash \{a, b\}$.

Friends-and-strangers graphs generalize many other objects, see their paper for details.

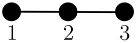
We may assume $V(X) = V(Y) = [n]$. In this case, the vertex set of $\mathsf{FS}(X, Y)$ is all permutations on $[n]$.

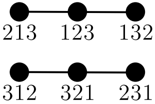
The Definition

The friends-and-strangers graph $\mathsf{FS}(X, Y)$ can be interpreted as follows. View $V(X)$ as n positions and $V(Y)$ as n people. Two people are friends iff they are adjacent in Y and two positions are adjacent iff they are adjacent in X. A bijection from $V(X)$ to $V(Y)$ represents n people standing on these n positions such that each person stands on precisely one position. At any point of time, two people can swap their positions iff they are friends and the two positions they stand are adjacent.

A natural question is how various configurations can be reached from other configurations when multiple such swaps are allowed. This is precisely the information that is encoded in $\mathsf{FS}(X, Y)$. Note that the components of $\mathsf{FS}(X, Y)$ are the equivalence classes of mutually-reachable (by the multiple swaps described above) configurations, so the connectivity is the basic aspect of interest in friends-and-strangers graphs.

An example

If $X = Y =$ 

, then $\mathsf{FS}(X, Y) =$ 

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Basic Properties

- The number of vertices $v(\mathsf{FS}(X,Y)) = n!$. The number of edges $e(\mathsf{FS}(X,Y)) = e(X)e(Y)(n-2)!$.
- The graph $\mathsf{FS}(X,Y)$ is isomorphic to $\mathsf{FS}(Y,X)$.
- The graph $\mathsf{FS}(X,Y)$ is bipartite, with at least two parts consisting of odd and even permutations.
- If one of X and Y is disconnected, or both of X and Y have cut vertices, then $\mathsf{FS}(X,Y)$ is disconnected.
- If X is a spanning subgraph of X' and Y is a spanning subgraph of Y', then $\mathsf{FS}(X,Y)$ is a spanning subgraph of $\mathsf{FS}(X',Y')$.

Previous results on the connectedness

The questions and results in literature on the connectedness of friends-and-strangers graphs roughly falls in three types

- The connectedness of $\mathsf{FS}(X, Y)$ when at least one of X, Y are specific graphs.
- The connectedness of $\mathsf{FS}(X, Y)$ when none of X, Y is specific graph.
- The connectedness of $\mathsf{FS}(X, Y)$ when both X and Y are random graphs.

The first result we will talk about belongs to the first type. And the second result we will talk about belongs to the third type.

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Previous Results

Theorem (Godsil-Royle, Algebraic Graph Theory (2001))

The graph $\mathsf{FS}(\mathsf{Complete_n}, \mathsf{Y})$ *is connected iff* Y *is connected.*

Theorem (Defant-Kravitz, Comb. Theory 1 (2021))

The graph $\mathsf{FS}(\mathsf{Path}_n, Y)$ *is connected iff* Y *is complete.*

Let $2 \leq k \leq n$ be integers. A lollipop $\mathsf{Lollipop}_{n-k,k}$ is a graph of order n obtained by identifying one end of a path of order $n-k+1$ with a vertex of a complete graph of order k.

It is easy to see that $\mathsf{Lollipop}_{0,n} = \mathsf{Complete_n}$ and $\mathsf{Lollipop}_{n-2,2} = \mathsf{Path}_n$, that is, the lollipop graph entends both path graph and complete graph. So, it is natural to ask for a sufficient and necessary condition for $\mathsf{FS}(\mathsf{Lollipop_{n-k,k}}, \mathsf{Y})$ to be connected, which entends the above two theorems.

Previous Results

Defant and Kravitz started to study the connectedness of FS(Lollipop$_{n-k,k}$, Y) and showed the following.

Theorem (Defant-Kravitz, Comb. Theory 1 (2021))

If FS(Lollipop$_{\mathsf{n-k,k}}$, Y) *is connected, then* $\delta(Y) \geq n-k+1$.

Theorem (Defant-Kravitz, Comb. Theory 1 (2021))

The graph FS(Lollipop$_{\mathsf{n-3,3}}$, Y) *is connected if and only if* $\delta(Y) \geq n-2$.

Theorem (Defant-Kravitz, Comb. Theory 1 (2021))

The graph FS(Lollipop$_{\mathsf{n-5,5}}$, Y) *is connected if* $\delta(Y) \geq n-3$, *and* FS(Lollipop$_{\mathsf{n-5,5}}$, Y) *is disconnected if* $\delta(Y) \leq n-5$.

Our Results

We give a sufficient and necessary condition for $\mathsf{FS}(\mathsf{Lollipop}_{\mathsf{n-k,k}}, \mathsf{Y})$ to be connected.

Theorem (Wang-Chen 2022+)

The graph $\mathsf{FS}(\mathsf{Lollipop}_{\mathsf{n-k,k}}, \mathsf{Y})$ *is connected if and only if every* k*-vertex induced subgraph of* Y *is connected, which is equivalent to* Y *is* $(n-k+1)$*-connected.*

Corollary

Let Y *be a graph on* n *vertices with* $\delta(Y) = n-4$*. Then the graph* $\mathsf{FS}(\mathsf{Lollipop}_{n-5,5}, Y)$ *is connected if and only if* Y *does not contain any induced subgraph isomorphic to* $\mathsf{Complete}_3 \cup \mathsf{Path}_2$ *or* $\mathsf{Path}_3 \cup \mathsf{Path}_2$.

Our idea comes from a paper written by Defant, Dong, Lee and Wei, in which they studied the case when X is a tree.

Proof Sketch: Connected Part

Proposition

Let $2 \leq k \leq n$ be integers and Y be a graph on n vertices. If every k-vertices induced subgraph of Y is connected, then $\mathsf{FS}(\mathsf{Lollipop}_{n-k,k}, \mathsf{Y})$ is connected.

Lemma (Defant-Dong-Lee-Wei, 2022+)

Let X and Y be connected graphs on n vertices with $\Delta(X) = k \geq 2$. Suppose every induced subgraph of Y with k vertices is connected. Let σ be a vertex of $\mathsf{FS}(X, Y)$, and fix $x \in V(X)$ and $y \in V(Y)$. Then there exists a vertex σ' in the same component of $\mathsf{FS}(X, Y)$ as σ such that $\sigma'(x) = y$.

Proof Sketch: Connected Part

The proof is by proceeding induction on n. The base case $n = k$ is guaranteed by the connectedness of $\mathsf{FS}(\mathsf{Complete}_\mathsf{n}, \mathsf{Y})$. We can reduce the case that the number of vertices is n into $n-1$ by using the above lemma since $\mathsf{Lollipop}_{\mathsf{n-k,k}} - \{1\}$ is isomorphic to $\mathsf{Lollipop}_{\mathsf{n-k-1,k}}$.

Proof Sketch: Disconnected Part

Proposition

If there exists a disconnected induced subgraph Y_0 of Y with k vertices, then $\mathsf{FS}(\mathsf{Lollipop}_{n-k,k}, \mathsf{Y})$ *is disconnected.*

Proof.

Let $V(Y_0) = A \cup B$. We say a vertex σ of $\mathsf{FS}(\mathsf{Lollipop}_{n-k,k}, \mathsf{Y})$ is *special* if there exists an $a_0 \in A$ such that $\sigma^{-1}(a_0) \in [n-k+1]$ and $\sigma^{-1}(B) = \{\sigma^{-1}(y) \mid y \in B\} \cap \{1, 2, \ldots, \sigma^{-1}(a_0)\} = \varnothing$. Any special vertex is not adjacent to any non-special vertex, which can be proven by arguing on several cases.

□

Open Problem

Let $2 \leq k \leq n$ be integers. The *dandelion* graph $\text{Dand}_{n-k,k}$ is obtained by identifying one end of a $\mathsf{Path}_{\mathsf{n-k+1}}$ with the center of a $\mathsf{Star}_{\mathsf{k}}$. The graph $\text{Dand}_{n-2,2}$ is $\text{Lollipop}_{n-2,2}$, and $\text{Dand}_{n-k,k}$ is a proper spanning subgraph of $\text{Lollipop}_{n-k,k}$ for $k \geq 3$.

Defant and Kravitz showed that $\text{FS}(\text{Lollipop}_{n-3,3}, Y)$ is connected if and only if $\text{FS}(\text{Dand}_{n-3,3}, Y)$ is connected. This leads us to ask when the edges not in the spanning subgraph $\text{Dand}_{n-k,k}$ of $\text{Lollipop}_{n-k,k}$ are not necessary for the connectedness of $\text{FS}(\text{Lollipop}_{n-k,k}, Y)$. More precisely, we have the following.

Open Problem

Problem

For what k and n, it holds that $\mathrm{FS}(\mathrm{Lollipop}_{n-k,k}, Y)$ *is connected if and only if* $\mathrm{FS}(\mathrm{Dand}_{n-k,k}, Y)$ *is connected?*

By our theorem and a result due to Defant, Dong, Lee and Wei, the statement holds for $n \geq 2k - 1$. On the other hand, by the connectedness of $\mathrm{FS}(\mathrm{Complete}_n, \mathrm{Y})$ and a result of Wilson, the statement is false for $n = k$.

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Previous results

Alon, Defant, and Kravitz showed that the threshold probability guaranteeing the connectedness of $\mathsf{FS}(X, Y)$ is $p_0 = n^{-1/2+o(1)}$.

Theorem (Alon, Defant, and Kravitz, JCTB (2022))

Fix some small $\varepsilon > 0$, and let X and Y be two graphs independently chosen from $\mathcal{G}(n, p)$, where $p = p(n)$ depends on n. If

$$p \leq \frac{2^{-1/2} - \varepsilon}{n^{1/2}},$$

then $\mathsf{FS}(X, Y)$ is disconnected w.h.p. If

$$p \geq \frac{\exp(2(\log n)^{2/3})}{n^{1/2}},$$

then $\mathsf{FS}(X, Y)$ is connected w.h.p.

Previous results

They suggested to investigate the general asymmetric situation, that is, $X \in \mathcal{G}(n, p_1)$ and $Y \in \mathcal{G}(n, p_2)$. We determines, up to a sub-polynomial factor building off of their work, a threshold condition for when $\mathsf{FS}(X, Y)$ is connected.

Our results

Theorem

Fix some small $\varepsilon > 0$, and let X and Y be independently chosen random graphs in $\mathcal{G}(n, p_1)$ and $\mathcal{G}(n, p_2)$, respectively, where $p_1 = p_1(n)$ and $p_2 = p_2(n)$ depend on n. Let $p_0 = \frac{\exp(2(\log n)^{2/3})}{n^{1/2}}$. If either

$$p_1 p_2 \leq \frac{(1-\varepsilon)/2}{n} \quad \textit{and} \quad p_1, p_2 \gg \frac{\log n}{n},$$

or

$$\min\{p_1, p_2\} \leq \frac{\log n + c(n)}{n} \textit{ for some } c(n) \to -\infty,$$

then $\mathsf{FS}(X, Y)$ *is disconnected w.h.p. If*

$$p_1 p_2 \geq p_0^2 \quad \textit{and} \quad p_1, p_2 \geq \frac{2}{(\log n)^{1/3}} p_0,$$

then $\mathsf{FS}(X, Y)$ *is connected w.h.p.*

Proof sketch: a sufficient condition for connectedness

Our proof idea comes from Alon, Defant, and Kravitz. The disconnected part is easy to deduce. For the connected part, a sufficient condition for connectedness is needed.

We say u and v are (X, Y)-*exchangeable from* σ if σ and $\tau_{uv} \circ \sigma$ are in the same component, where $\tau_{uv} : V(Y) \mapsto V(Y)$ is the bijection such that $\tau_{uv}(u) = v$, $\tau_{uv}(v) = u$ and $\tau_{uv}(w) = w$ for any $w \in V(Y) \backslash \{u, v\}$.

Lemma (Alon, Defant, and Kravitz, JCTB (2022))

Let X, Y be two graphs on n vertices, and X is connected. Suppose for any two vertices $u, v \in Y$ and every σ satisfying $\{\sigma^{-1}(u), \sigma^{-1}(v)\} \in E(X)$, the vertices u and v are (X, Y)-exchangeable from σ. Then FS(X, Y) *is connected.*

In general, it is not easy to know if two vertices $u, v \in V(Y)$ are (X, Y)-exchangeable from some bijection $\sigma : V(X) \mapsto V(Y)$. So, the embeddings of small graphs are considered.

Proof sketch: the embedding of small graphs

Let G and H be two graphs on vertex set $[m]$, and $\sigma : V(X) \mapsto V(Y)$ be a bijection. Let $V_1, \ldots, V_m$ be a list of m pairwise disjoint sets of vertices of Y. We say that the pair of graphs (G, H) is *embeddable in* (X, Y) *w.r.t. the sets* $V_1, \ldots, V_m$ *and the bijection* σ if there exist vertices $v_i \in V_i$ for all $i \in [m]$ such that for all $i, j \in [m]$, we have

$$\{i,j\} \in E(H) \Rightarrow \{v_i, v_j\} \in E(Y) \text{ and}$$
$$\{i,j\} \in E(G) \Rightarrow \{\sigma^{-1}(v_i), \sigma^{-1}(v_j)\} \in E(X).$$

Suppose $q_1, \ldots, q_m$ are nonnegative integers satisfying $q_1 + \cdots + q_m \leq n$. We say (G, H) is $(q_1, \ldots, q_m)$*-embeddable in* (X, Y) if (G, H) is embeddable in (X, Y) w.r.t. every list $V_1, \ldots, V_m$ of pairwise disjoint subsets of $V(Y)$ satisfying $|V_i| = q_i$ for all $i \in [m]$ and every bijection $\sigma : V(X) \mapsto V(Y)$.

Proof sketch: the embedding lemma for embeddings

The following lemma deals with when we can embed a pair of small graphs into a pair of large random graphs.

Lemma

Let $m, n, q_1, \ldots, q_m$ *be positive integers such that* $Q = q_1 + \cdots + q_m \leq n$, *and* G, H *be two graphs on the vertex set* $[m]$. *Let* X *and* Y *be independently chosen random graphs in* $\mathcal{G}(n, p_1)$ *and* $\mathcal{G}(n, p_2)$, *respectively, where* $p_1 = p_1(n)$ *and* $p_2 = p_2(n)$ *depend on* n. *If for every set* $J \subseteq [m]$ *satisfying* $|E(G|_J)| + |E(H|_J)| \geq 1$ *we have*

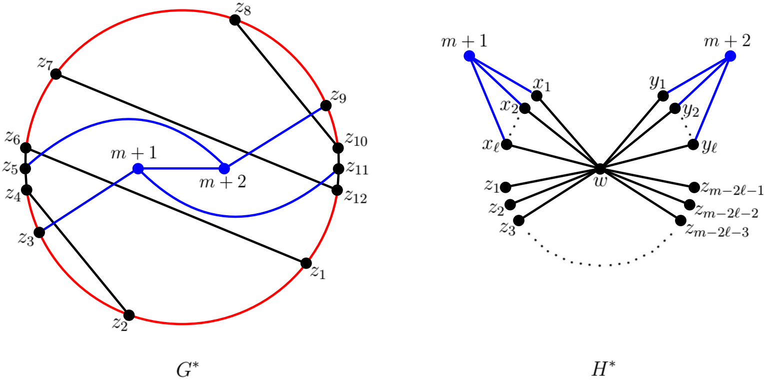
$$p_1{}^{|E(G|_J)|} p_2^{|E(H|_J)|} \prod_{j \in J} q_j \geq 3 \cdot 2^{m+1} Q \log n,$$

then the probability that the pair (G, H) *is* $(q_1, \ldots, q_m)$*-embeddable in* (X, Y) *is at least* $1 - n^{-Q}$.

We used Janson Inequality to prove this lemma.

Proof sketch: the two graphs we used to embed

We embed two graphs constructed by Alon, Defant and Kravitz, where the vertices $m+1$ and $m+2$ are (G^*, H^*)-exchangeable from Id.



If we can constructed more graphs with good properties, then the above lemma can by applied to deduce more results.

Thank you very much!