The Betti Number of the Independence Complex of Ternary Graphs

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Overview

Introduction

- $f_G = \sum(-1)^{|A|}$ over all independent sets A.
- $I(G)$: the simplicial complex whose faces are the independent sets of $V(G)$.
- $\tilde{b}_i(I(G)) = dim\tilde{H}_i(I(G))$: the i-th reduced Betti number of $I(G)$.
- $b(G) = \sum_i \tilde{b}_i(I(G))$.
- Homological fact: $\chi_{I(G)} = 1 + \sum_i (-1)^i \tilde{b}_i(G) = \sum(-1)^{|A|-1}$, sum over all the non-empty independent sets in G.
- $f_G = \sum_{i=0}^{\infty}(-1)^{i+1}\tilde{b}_i(G)$
- $|f_G| \leq b(G)$
- A graph is *ternary* if it has no induced cycle of length divisible by three.

Introduction

Theorem(Chudnovsky, Scott, Seymour and Spirkl, 2020)

If G is a graph with no induced cycle of length divisible by three, then $|f_G| \leq 1$.

Theorem(Hehui Wu and Wentao Zhang, 2021)

If G is a graph with no induced cycle of length divisible by three, then $b(G) \leq 1$.

Theorem(Engstrom, 2020)

If G is a graph without cycles of length divisible by three, then $I(G)$ is contractible or homotopy equivalent to a sphere.

Conjecture(Engstrom, 2020)

If G is a graph without induced cycles of length divisible by three, then $I(G)$ is contractible or homotopy equivalent to a sphere.

Some Notation

- Given a graph G, X is an independent set of G and Y is a vertex set disjoint from X.
- $G(X \mid Y)$ is the subgraph induced by $V(G) - N[X] - Y$.
- Use $I(X \mid Y)$, $b(X \mid Y)$ and $\tilde{b}_i(X \mid Y)$ for Simplicity.

$$I(X \mid Y) = I(G(X \mid Y))$$

$$b(X \mid Y) = b(G(X \mid Y))$$

$$\tilde{b}_i(X \mid Y) = \tilde{b}_i(I(G(X \mid Y)))$$

Mayer-Vietoris Sequence

Let K' and K'' be subcomplexes such that $K = K' \cup K''$ and let $L = K' \cap K''$. There is an exact sequence of reduced homology groups called the Mayer-Vioteris sequence.

$$\cdots \longrightarrow \tilde{H}_i(L) \xrightarrow{\lambda_i} \tilde{H}_i(K') \oplus \tilde{H}_i(K'') \longrightarrow \tilde{H}_i(K) \longrightarrow \tilde{H}_{i-1}(L) \xrightarrow{\lambda_{i-1}} \tilde{H}_{i-1}(K') \oplus \tilde{H}_{i-1}(K'') \longrightarrow \cdots \longrightarrow \tilde{H}_0(K) \longrightarrow 0$$

Short exact sequence:

$$0 \longrightarrow \operatorname{cok} \lambda_i \longrightarrow \tilde{H}_i(K) \longrightarrow \ker \lambda_{i-1} \longrightarrow 0$$

Mayer-Vietoris Sequence

- $N_i = \ker \lambda_i$, $\beta(N_i)$ is its dimension.

$$\begin{aligned}\beta_i(K) &= \beta(\operatorname{cok} \lambda_i) + \beta(\ker \lambda_{i-1}) \\ &= \beta(\tilde{H}_i(K') \oplus \tilde{H}_i(K'')/\operatorname{im} \lambda_i) + \beta(N_{i-1}) \\ &= \beta_i(K') + \beta_i(K'') - \beta_i(L) + \beta(N_i) + \beta(N_{i-1})\end{aligned} \tag{1}$$

- Suppose v is a vertex of G, take $K = I(G)$, $K' = I(G - v)$, $K'' = I(G - N(v))$ and $L = I_G(v \,|\, \emptyset)$.
- If H has an isolated vertex, then $b(H) = 0$.

$$\tilde{b}_i(G) = \tilde{b}_i(\emptyset \,|\, v) - \tilde{b}_i(v \,|\, \emptyset) + \beta(N_i) + \beta(N_{i-1}), \quad \forall i. \tag{2}$$

$$\beta(N_i) \le \tilde{b}_i(v \,|\, \emptyset), \quad \forall i. \tag{3}$$

Proof of Main Theorem

Theorem

If $b(G) \geq 2$ and $b(H) \leq 1$ for every induced subgraph H of G, then $G = C_{3k}$ for some integer k.

- $b(X \mid Y) = 0$ or 1 if $X \cup Y \neq \emptyset$.
- If $b(H) = 1$ for some graph H, let $d(H)$ be the dimension of the reduced Betti number taking value 1. $d(H) = i$ if $\tilde{b}_i(H) = 1$ and $\tilde{b}_j(H) = 0$ for $j \neq i$.
- If $b(H) = 0$, let $d(H)$ be '$*$'.
- If X is not independent, let $d(X \mid Y) = *$.
- If H is null graph, let $d(H) = -1$.

Proof of Main Theorem

Lemma 1

For any disjoint vertex set X and Y in G with $X \cup Y \neq \emptyset$ and a vertex v not in X or Y, the triple $(d(X \mid Y), d(v), d(v))$ fits into one of the following four patterns: $(k, *, k)$, $(*, *, *)$, $(*, k, k)$ and $(k+1, k, *)$ for some integer k.

(X, Y)	k	$*$	$*$	$k+1$
$(X \cup \{v\} \mid Y)$	$*$	$*$	k	k
$(X \mid Y \cup \{v\})$	k	$*$	k	$*$

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(X, Y)		k		$*$		$*$		$k+1$	
$(X \cup \{v\} \mid Y)$	$(X \mid Y \cup \{v\})$	$*$	k	$*$	$*$	k	k	k	$*$

Proof of Main Theorem

Lemma 2

Suppose X, Y are vertex set of G with $d(X \mid Y) = k$ for some integer k. If v_1, v_2 are two vertices not in $X \cup Y$ with $d(v_1\} = k - 1$ and $d(v_2) = *$, then $d(v_1, v_2) = *$.

Proof of Main Theorem

Lemma 3

There is some $k \geq 0$ such that $\tilde{b}_k(G) = 2$ and $\tilde{b}_i(G) = 0$ for all $i \neq k$. Furthermore, for every vertex v, $d(v \mid \emptyset) = k - 1$ and $d(\emptyset \mid v) = k$.

Proof of Main Theorem

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$(|X|, |Y|)$:

$$\begin{array}{cccccc}
 & & (1,0) & (0,1) & & \\
 & (2,0) & (1,1) & (0,2) & & \\
(3,0) & (2,1) & (1,2) & (0,3) & & \\
 & & \vdots & & & \\
(t,0) & (t-1,1) & \cdots & \cdots & (1,t-1) & (0,t)
\end{array}$$

$d(X \mid Y)$:

$$\begin{array}{cccccc}
k-1 & / & & & & \\
k-2 & * & / & & & \\
k-3 & * & * & / & & \\
 & & \vdots & & & \\
k-t & * & \cdots & \cdots & * & /
\end{array}$$

Proof of Main Theorem

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$(|X|, |Y|)$:

$$
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 & (2,0) & (1,1) & (0,2) & & \\
(3,0) & (2,1) & (1,2) & (0,3) & & \\
 & & \vdots & & & \\
(t,0) & (t-1,1) & \cdots & \cdots & (1,t-1) & (0,t)
\end{array}
$$

$d(X \mid Y)$:

$$
\begin{array}{cccccc}
k-1 & l & & & & \\
k-2 & * & l & & & \\
k-3 & * & * & l & & \\
 & & \vdots & & & \\
k-t & * & \cdots & \cdots & * & l
\end{array}
$$

Proof of Main Theorem

Construct a new graph H on $V(G)$ such that u, v are adjacent if and only if $d(u, v \mid \emptyset) = k - 2$.

Proposition 4

In H, any two vertices u, v satisfies

1. If $u \sim v$ in H, then $u \nsim v$ in G. That is, $E(G) \cap E(H) = \emptyset$.
2. If $u \sim v$ in H, then $d(u, v \mid \emptyset) = k - 2$, $d(u \mid v) = d(v \mid u) = *$, and $d(\emptyset \mid u, v) = k$,
3. If $u \nsim v$ in H, then $d(u, v \mid \emptyset) = d(\emptyset \mid u, v) = *$, and $d(u \mid v) = d(v \mid u) = k - 1$.

Proof of Main Theorem

Lemma 5

Every component C of H is a complete graph. Furthermore, for any disjoint subsets X and Y of $V(C)$ with $X \cup Y \neq \emptyset$, we have

$$d(X \mid Y) = \begin{cases} k - |X|, & Y = \emptyset, \\ *, & X, Y \neq \emptyset, \\ k, & X = \emptyset. \end{cases}$$

Proof of Main Theorem

Lemma 5

Every component C of H is a complete graph. Furthermore, for any disjoint subsets X and Y of $V(C)$ with $X \cup Y \neq \emptyset$, we have

$$d(X \mid Y) = \begin{cases} k - |X|, & Y = \emptyset, \\ *, & X, Y \neq \emptyset, \\ k, & X = \emptyset. \end{cases}$$

$(|X|, |Y|)$:

$$\begin{array}{cccccc} & & (1,0) & (0,1) & & \\ & (2,0) & (1,1) & (0,2) & & \\ (3,0) & (2,1) & (1,2) & (0,3) & & \\ & & \vdots & & & \\ (t,0) & (t-1,1) & \cdots & \cdots & (1,t-1) & (0,t) \end{array}$$

$d(X, Y)$:

$$\begin{array}{cccccc} & & k-1 & k & & \\ & k-2 & * & k & & \\ k-3 & * & * & k & & \\ & & \vdots & & & \\ k-t & * & \cdots & \cdots & * & k \end{array}$$

Proof of Main Theorem

Claim 6

There does not exist a vertex v with all neighbors in G located in one component of H.

Proof of Main Theorem

Lemma 7

There do not exist two edges v_1v_2, v_3v_4 in G, with v_1, v_2, v_3, v_4 located in four distinct components of H.

$(|X|, |Y|)$:

$$\begin{array}{ccccc} & & (1,0) & (0,1) & \\ & (2,0) & (1,1) & (0,2) & \\ (3,0) & (2,1) & (1,2) & (0,3) & \\ (4,0) & (3,1) & (2,2) & (1,3) & (0,4) \end{array}$$

$d(X \mid Y)$:

$$\begin{array}{cccc} & & k-1 & k \\ & * & k-1 & * \\ * & * & k-1 & k-1 \\ * & * & * & ? \end{array}$$

Proof of Main Theorem

Theorem

If $b(G) \geq 2$ *and* $b(H) \leq 1$ *for every induced subgraph* H *of* G*, then* $G = C_{3k}$ *for some integer* k*.*

The End