The Minimum Number of Clique-Saturating Edges

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June 10, 2022

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Notations

1. For an integer $n \geq 1$, denote by [n] the set $\{1, 2, ..., n\}$.
2. All graphs considered are **finite**, **undirected** and **simple**.
3. Let $G[A]$ denote the subgraph **induced** on vertex set A, i.e. $E(G[A])$ consists of all edges in $E(G)$ with both endpoints in A.
4. For any vertex subset $U \subseteq V(G)$, denote $N(U) := \cap_{v \in U} N(v)$.

Notations

1. A **complete graph** on t vertices, denoted by K_t, is a graph in which every pair of vertices forms an edge.
2. A **complete bipartite graph** on vertex set $X \cup Y$, denoted by $K_{|X|,|Y|}$, is a graph in which two vertices form an edge if and only if one of them is in X and the other one is in Y.
3. A graph $G = (V, E)$ is r-**partite** if the vertex set V can be partitioned into r disjoint sets $V_1, V_2, ..., V_r$ such that each V_i, $1 \leq i \leq r$, is an independent set.
4. The **blow-up** of a graph is obtained by replacing every vertex with a finite collection of copies so that the copies of two vertices are adjacent if and only if the originals are.

Notation: Turán numbers

- We say that G is H-**free** if G does not contain H as a subgraph.

Definition 1.1

The *Turán number* of H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges an n-vertex H-free graph can have. And let $\mathrm{EX}(n, H)$ denote the set of those n-vertex H-free graph(s) with $\mathrm{ex}(n, H)$ edges.

Notation: Turán graphs

Definition 1.2

The unique complete p-partite graphs on $n \geq p$ vertices whose partition sets differ in size by at most 1 are called *Turán graphs*; we denote them by $T_p(n)$ and their number of edges by $t_p(n)$. For all $n \leq p$, $T_p(n) = K_n$.

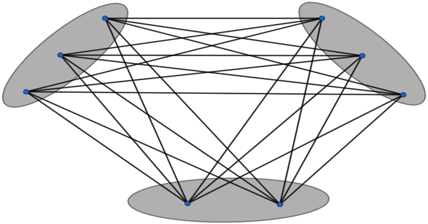

Figure 1. $T_3(8)$.

Theorem: Turán number for cliques

Theoerm 1.3 (Mantel, 1907)

$\mathrm{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$ and $\mathrm{EX}(n, K_3) = \{T_2(n)\}$.

Theorem 1.4 (Turán, 1941)

For all integers $p \geq 2$,

$$\mathrm{ex}(n, K_{p+1}) = t_p(n)$$

and

$$\mathrm{EX}(n, K_{p+1}) = \{T_p(n)\}$$

Introduction: Clique-saturating edges

Definition 2.1

For $p \geq 2$, let G be a K_{p+1}-free graph and e be a *non-edge* of G (i.e., an edge in the complement of G). We say e is a *K_{p+1}-saturating edge of G*, if $G + e$ contains a copy of K_{p+1}.

- Note that a K_{p+1}-free graph G is maximal if and only if every non-edge of G is a K_{p+1}-saturating edge (let us call this property $\star$).
- So in other words, Turán's Theorem determines the maximum number of edges $e(G)$ over all K_{p+1}-free graphs G satisfying the property $\star$.

- On the other hand, Zykov (1949) and independently Erdős, Hajnal and Moon (1964) determined the minimum number $e(G)$ over all n-vertex K_{p+1}-free graphs G satisfying the property $\star$.

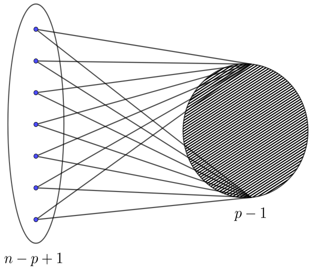

Figure 2. The n-vertex complement graph of K_{n-p+1}.

Introduction: K_{p+1}-saturating edges

Definition 2.2

For a K_{p+1}-free graph G, let $f_{p+1}(G)$ denote the number of K_{p+1}-saturating edges of G. Let $f_{p+1}(n,m)$ be the minimum number of K_{p+1}-saturating edges of an n-vertex K_{p+1}-free graph with m edges.

- Note that, for $0 \leq m \leq \mathrm{ex}(n, K_{p+1}) - 1$,

$$f_{p+1}(n, m+1) \geq f_{p+1}(n, m).$$

- By Turán's theorem, we also have

$$f_{p+1}(n, \mathrm{ex}(n, K_{p+1})) = \binom{n}{2} - \mathrm{ex}(n, K_{p+1}) \sim \frac{n^2}{2p}.$$

Properties of $f_{p+1}(n, m)$

- Moreover, for all integers $p \geq 3$, the example of the Turán graph $T_{p-1}(n)$ shows that

$$f_{p+1}(n, m) = 0 \quad \text{for all} \;\; 0 \leq m \leq \mathrm{ex}(n, K_p).$$

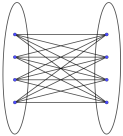

Figure 3. The Turán graph $T_2(n)$.

- What is the value of $f_{p+1}(n, \mathrm{ex}(n, K_p) + 1)$?

Erdős and Tuza's Conjecture on K_4-saturating edges

- Erdős and Tuza (1990) proved that $f_4(n, \lfloor \frac{n^2}{4} \rfloor + 1) \geq cn^2$ for some constant $c > 0$. And they also made the following conjecture.

Conjecture 2.3 (Erdős and Tuza, 1990).

$$f_4\left(n, \left\lfloor \frac{n^2}{4} \right\rfloor + 1\right) = (1 + o(1))\, \frac{n^2}{16}.$$

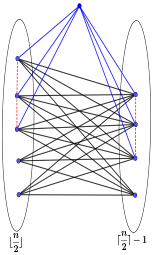

Figure 4. A K_4-free graph H with $e(H) = \lfloor \frac{n^2}{4} \rfloor + 1$ and $f_4(H) = (1 + o(1))\, \frac{n^2}{16}$.

Balogh and Liu's Theorem on K_4-saturating edges

- This however was disproved by Balogh and Liu (2014), where they constructed an n-vertex K_4-free graph with $\lfloor \frac{n^2}{4} \rfloor + 1$ edges and with only $(1 + o(1))\frac{2n^2}{33}$ K_4-saturating edges.

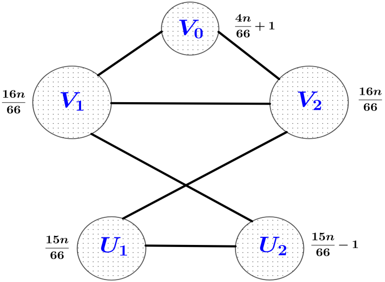

Figure 5. A K_4-free graph H with $e(H) = n^2/4 + n/66$ and $f_4(H) = 2n^2/33 - 7n/33$.

Balogh and Liu's Theorem on K_4-saturating edges

- Furthermore, Balogh and Liu showed that the above construction is best possible.

Theorem 2.4 (Balogh and Liu, 2014).

$f_4(n, \lfloor \frac{n^2}{4} \rfloor + 1) = (1 + o(1)) \frac{2n^2}{33}$.

Balogh and Liu's Conjecture on K_{p+1}-saturating edges

- Balogh and Liu also made an explicit conjecture for general p suggested by a natural generalization of their K_4-free construction.

Conjecture 2.5 (Balogh and Liu, 2014).

For all integers $p \geq 3$,

$$f_{p+1}(n, \mathrm{ex}(n, K_p) + 1) = \left(\frac{2(p-2)^2}{p(4p^2 - 11p + 8)} + o(1) \right) n^2.$$

Our results on K_{p+1}-saturating edges

- The main result of our paper is to prove the above conjecture of Balogh and Liu.

Theorem 3.1 (H., Ma, Ma and Ye, 2022^+).

For all integers $p \geq 3$, $f_{p+1}\big(n, \mathrm{ex}(n, K_p) + 1\big) = \left(\frac{2(p-2)^2}{p(4p^2-11p+8)} + o(1)\right) n^2$.

Our results on K_{p+1}-saturating edges

- Most of the paper will be devoted to the lower bound of the following theorem. Note that for any integer $p \geq 3$, $f_{p+1}(G) = 0$ holds for $G = T_{p-1}(n)$.

Theorem 3,2 (H., Ma, Ma and Ye, 2022$^+$).

Let $p \geq 3$ and $n \geq 8p^5$ be integers. Let $\mathcal{G}$ be the family consisting of all n-vertex K_{p+1}-free graphs with exactly $\mathrm{ex}(n, K_p)$ edges. Then

$$\min_{G \in \mathcal{G} \setminus \{T_{p-1}(n)\}} f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2-11p+8)} n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8} n + O_p(1).$$

In addition, if n is divisible by $p(p-1)(4p^2-11p+8)$, then

$$\min_{G \in \mathcal{G} \setminus \{T_{p-1}(n)\}} f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2-11p+8)} n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8} n.$$

The proof ideas of Theorems 3.1 and 3.2

Theorem 3.1 (H., Ma, Ma and Ye, 2022$^+$).

For all integers $p \geq 3$, $f_{p+1}\left(n, \mathrm{ex}(n, K_p) + 1\right) = \left(\frac{2(p-2)^2}{p(4p^2-11p+8)} + o(1)\right) n^2$.

Theorem 3,2 (H., Ma, Ma and Ye, 2022$^+$).

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In addition, if n is divisible by $p(p-1)(4p^2-11p+8)$, then

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The constructions for the upper bounds

These graphs are suggested by Balogh and Liu, each of which is an appropriate blow-up of the following graph: take a complete $(p-1)$-partite graph $K = K_{2,\ldots,2}$ and add a new vertex by making it adjacent to exactly one vertex in each partite set of K.

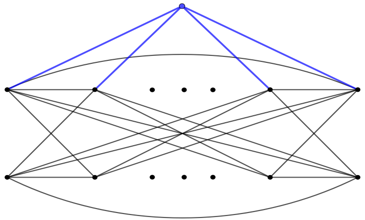


Figure 6. The graph used to construct the upper bounds.

The upper bound of Theorem 3.2

In the rest of this section, for convenience, we assume that $n = p(p-1)(4p^2 - 11p + 8)x$.

We will construct an n vertices, K_{p+1}-free graph H_0, with exactly $\mathrm{ex}(n, K_p)$ edges and

$$f_{p+1}(H_0) = \frac{2(p-2)^2}{p(4p^2 - 11p + 8)} n^2 - \frac{(p-2)(2p-3)}{p(4p^2 - 11p + 8)} n.$$

The upper bound of Theorem 3.2

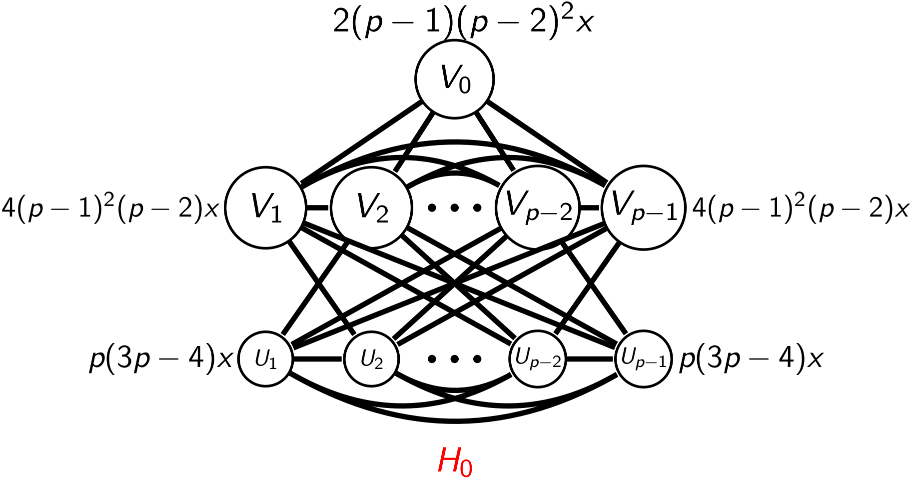


Figure 7. Constructions for the upper bounds of Theorems 3.1 and 3.2.

The upper bound of Theorem 3.2

We can check that H_0 is K_{p+1}-free on $n = p(p-1)(4p^2 - 11p + 8)x$ vertices with $\mathrm{ex}(n, K_p) = \frac{p-2}{2(p-1)} \cdot p^2(p-1)^2(4p^2 - 11p + 8)^2 x^2$ edges.

The only K_{p+1}-saturating edges are the pairs in V_i for $0 \le i \le p-1$. This leads to

$$f_{p+1}(H_0) = \frac{2(p-2)^2}{p(4p^2 - 11p + 8)} n^2 - \frac{(p-2)(2p-3)}{4p^2 - 11p + 8} n,$$

completing the proof for the upper bound.

- The construction for the upper bound of Theorem 3.1 is quite similar to the one above. The only differences are the sizes of the parts in the blow-up.

Proof of Theorem 3.1

Proof.

In this section, assuming Theorem 3.2, we complete the proof of Theorem 3.1. It suffices to prove the lower bound. Let G be a K_{p+1}-free graph with $\mathrm{ex}(n, K_p) + 1$ edges. By Turán's Theorem, G contains a copy of K_p. Let G' be obtained from G by removing a single edge such that G' still contains a K_p. Then G' is K_{p+1}-free with $\mathrm{ex}(n, K_p)$ edges. As G' contains a K_p, it cannot be the Turán graph $T_{p-1}(n)$. By Theorem 3.2, we have

$$f_{p+1}(G) \geq f_{p+1}(G') \geq \frac{2(p-2)^2}{p(4p^2-11p+8)}n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8}n + O_p(1),$$

finishing the proof of Theorem 3.1. □

The proof of Theorem 3.2

Theorem 3,2 (H., Ma, Ma and Ye, 2022^+).

Let $p \geq 3$ and $n \geq 8p^5$ be integers. Let $\mathcal{G}$ be the family consisting of all n-vertex K_{p+1}-free graphs with exactly $\mathrm{ex}(n, K_p)$ edges. Then

$$\min_{G \in \mathcal{G} \setminus \{T_{p-1}(n)\}} f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2-11p+8)} n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8} n + O_p(1).$$

In addition, if n is divisible by $p(p-1)(4p^2-11p+8)$, then

$$\min_{G \in \mathcal{G} \setminus \{T_{p-1}(n)\}} f_{p+1}(G) = \frac{2(p-2)^2}{p(4p^2-11p+8)} n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8} n.$$

Proof idea of the lower bound of Theorem 3.2

- Let G be any n-vertex K_{p+1}-free graph with $\mathrm{ex}(n, K_p)$ edges, but not the $(p-1)$-partite Turán graph $T_{p-1}(n)$.
- Here, for convenience, we assume that n is divisible by $p(p-1)(4p^2-11p+8)$.
- It suffices to show that $f_{p+1}(G)$ is bounded from below by the desired formula ($f_{p+1}(G) \geq \frac{2(p-2)^2}{p(4p^2-11p+8)}n^2 - \frac{(p-2)(2p-3)}{4p^2-11p+8}n$).

Proof idea of the lower bound of Theorem 3.2

- Following the approach of Balogh and Liu, we partition the vertex set of G into two parts $V(\mathcal{R})$ and its complement $V(G)\setminus V(\mathcal{R})$, where $\mathcal{R}$ is a maximum family of vertex-disjoint K_p's in G and $V(\mathcal{R})$ denotes the set of all vertices contained in $\mathcal{R}$.

Proof idea of the lower bound of Theorem 3.2

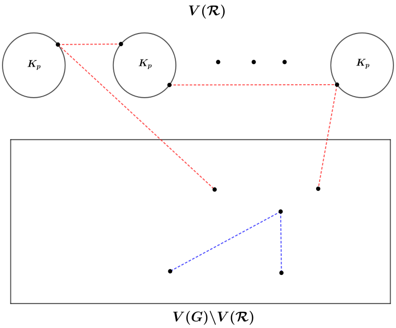
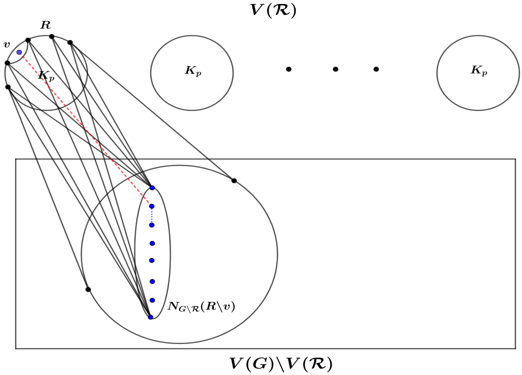


Figure 8. Two types of K_{p+1}-saturating edges of G.

- The problem is that when p is getting bigger, the complexity of computations based on these estimations will be difficult to handle.

Proof idea of the lower bound of Theorem 3.2

A key motivation for us comes after Lemma 4.4 (we will see later), which roughly says that for any p-clique R in $\mathcal{R}$, as long as there are enough edges between R and $V(G)\backslash V(\mathcal{R})$, any $p-1$ vertices of R have some common neighbors in $V(G)\backslash V(\mathcal{R})$ (it can even be set up as $\Omega(1)$ many if required).



Proof idea of the lower bound of Theorem 3.2

We now partition $V(G)$ into two parts $V(\mathcal{R})$ and $V(G)\setminus V(\mathcal{R})$ satisfying the following conditions

(i). $\mathcal{R}$ is a maximum family of vertex-disjoint K_p's in G, and

Proof idea of the lower bound of Theorem 3.2

We now partition $V(G)$ into two parts $V(\mathcal{R})$ and $V(G)\backslash V(\mathcal{R})$ satisfying the following conditions

(i). $\mathcal{R}$ is a maximum family of vertex-disjoint K_p's in G, and

(ii). subject to (i), the remaining graph $H_{\mathcal{R}} = G\backslash V(\mathcal{R})$ has the maximum number of edges.

Proof idea of the lower bound of Theorem 3.2

We now partition $V(G)$ into two parts $V(\mathcal{R})$ and $V(G)\setminus V(\mathcal{R})$ satisfying the following conditions

(i). $\mathcal{R}$ is a maximum family of vertex-disjoint K_p's in G, and

(ii). subject to (i), the remaining graph $H_{\mathcal{R}} = G\setminus V(\mathcal{R})$ has the maximum number of edges.

Let $|\mathcal{R}| := rn$. Since G contains a K_p, we have

$$1/n \leq r \leq 1/p. \tag{1}$$

Key Lemma

The following lemma is key in our proof. It shows that by the choice of $\mathcal{R}$ and $H_{\mathcal{R}}$, there are enough many edges incident to new p-cliques obtained from some $R \in \mathcal{R}$ by switching some vertices in R with vertices in $H_{\mathcal{R}}$ of equal size.

Lemma 4.1 (Key Lemma)

Let $R \in \mathcal{R}$ be a p-clique and C be a subclique of R. If there exists a clique C' in $H_{\mathcal{R}}$ of equal size as C such that $R' := (R \backslash C) \cup C'$ remains a clique in G, then $\mathcal{R}' := (\mathcal{R} \backslash \{R\}) \cup \{R'\}$ is also a maximum family of vertex-disjoint K_p's in G with $e(R', H_{\mathcal{R}'}) \geq e(R, H_{\mathcal{R}})$, where $H_{\mathcal{R}'} = G \backslash V(\mathcal{R}')$.

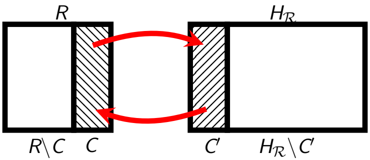


Figure 9. The proof of Lemma 4.1.

Proof of the key lemma

Proof.

First observe that $\mathcal{R}'$ is also a maximum family of rn vertex-disjoint K_p's. Let $H_{\mathcal{R}'} = G \backslash V(\mathcal{R}')$. So $H_{\mathcal{R}'} = (H_{\mathcal{R}} \backslash C') \cup C$ (see Figure 9). By (ii), we have $e(H_{\mathcal{R}}) \geq e(H_{\mathcal{R}'})$. Since $e(C') = e(C)$,

$$e(H_{\mathcal{R}}) = e(C') + e(C', H_{\mathcal{R}} \backslash C') + e(H_{\mathcal{R}} \backslash C') \text{ and}$$
$$e(H_{\mathcal{R}'}) = e(C) + e(C, H_{\mathcal{R}} \backslash C') + e(H_{\mathcal{R}} \backslash C'),$$

it follows that

$$e\left(C', H_{\mathcal{R}} \backslash C'\right) \geq e\left(C, H_{\mathcal{R}} \backslash C'\right).$$

Therefore, as $e(R \backslash C, C') = e(R \backslash C, C)$, one can derive that

$$e(R', H_{\mathcal{R}'}) - e(R, H_{\mathcal{R}}) = e\left(C', H_{\mathcal{R}} \backslash C'\right) - e\left(C, H_{\mathcal{R}} \backslash C'\right) \geq 0.$$

This completes the proof of Lemma 4.1. □

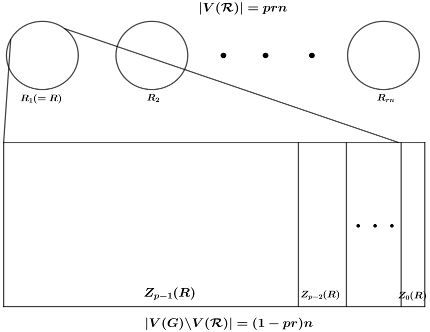
Proof idea of the lower bound of Theorem 3.2

For any p-clique $R \in \mathcal{R}$ and $0 \leq j \leq p$, we let

$$Z_j(R) = \{\text{all vertices in } H_{\mathcal{R}} \text{ that has exactly } j \text{ neighbors in } V(R)\}$$

$$\text{and } z_j(R) := |Z_j(R)|/n.$$

By the assumption that G is K_{p+1}-free, it is clear that $Z_p(R) = \emptyset$.

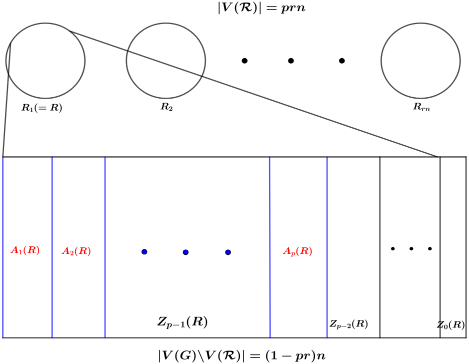


Proof idea of the lower bound of Theorem 3.2

We will also need to consider a refined partition of $Z_{p-1}(R)$ as follows. Let $\{v_1, v_2, \ldots, v_p\}$ represent the vertex set of a given p-clique $R \in \mathcal{R}$. For any $i \in [p]$, define

$$A_i(R) := N_{H_{\mathcal{R}}}(R \backslash \{v_i\})$$

to be the common neighborhood of $V(R) \backslash \{v_i\}$ in $V(H_{\mathcal{R}})$.



Proof idea of the lower bound of Theorem 3.2

Let us observe that $A_i(R)$'s are pairwise vertex-disjoint independent sets in $Z_{p-1}(R)$ (for otherwise $\left(\bigcup_i A_i(R)\right) \cup R$ would contain a copy of K_{p+1}, a contradiction to G is K_{p+1}-free). In particular, we have

$$\sum_{i=1}^{p} |A_i(R)|/n = z_{p-1}(R). \tag{2}$$

Proof idea of the lower bound of Theorem 3.2

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$$\sum_{i=1}^{p} |A_i(R)|/n = z_{p-1}(R). \tag{2}$$

- It is crucial to see that every non-edge inside each $A_i(R)$ is a K_{p+1}-saturating edge in $H_{\mathcal{R}} = G \setminus V(\mathcal{R})$.

Proof idea of the lower bound of Theorem 3.2

Next we give three technical lemmas and we should emphasize in advance that these lemmas hold for any family $\mathcal{R}$ solely satisfying the condition (i).

The first one says that for any family $\mathcal{R}$ satisfying the condition (i), there is a $R^* \in \mathcal{R}$ such that $e(R^*, H_{\mathcal{R}})$ and $z_{p-1}(R^*)$ are large.

Proof idea of the lower bound of Theorem 3.2

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The first one says that for any family $\mathcal{R}$ satisfying the condition (i), there is a $R^* \in \mathcal{R}$ such that $e(R^*, H_{\mathcal{R}})$ and $z_{p-1}(R^*)$ are large.

Lemma 4.2

Suppose that $\mathcal{R}$ is under the condition (i) and $H_{\mathcal{R}} = G \backslash V(\mathcal{R})$. Then there exists a p-clique $R^* \in \mathcal{R}$ such that

$$e(R^*, H_{\mathcal{R}}) \geq \left(\frac{p(p-2)}{p-1} - \frac{p(2p^2-4p+1)}{2(p-1)} r \right) n. \qquad (3)$$

Moreover, for any $R^* \in \mathcal{R}$ satisfying (3), it holds that

$$z_{p-1}(R^*) \geq \frac{p-2}{p-1} - \frac{p(2p-3)}{2(p-1)} r. \qquad (4)$$

Proof idea of the lower bound of Theorem 3.2

Denote by $\ell_1^{\mathcal{R}}$ the number of K_{p+1}-saturating edges incident to $V(\mathcal{R})$, and by $\ell_2^{\mathcal{R}}$ the number of K_{p+1}-saturating edges in $H_{\mathcal{R}}$. Obviously $f_{p+1}(G) = \ell_1^{\mathcal{R}} + \ell_2^{\mathcal{R}}$.

The lemma below gives a lower bound on $\ell_1^{\mathcal{R}}$, which in particular shows that Theorem 3.2 holds in case r is close to $1/p$.

Proof idea of the lower bound of Theorem 3.2

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The lemma below gives a lower bound on $\ell_1^{\mathcal{R}}$, which in particular shows that Theorem 3.2 holds in case r is close to $1/p$.

Lemma 4.3

Suppose that $\mathcal{R}$ is under the condition (i). Then

$$\ell_1^{\mathcal{R}} \geq \left(\frac{p-2}{p-1} r - \frac{p(p-2)}{2(p-1)} r^2 \right) n^2 - \frac{pr}{2} n.$$

Moreover, if $r > \frac{2(p-2)(2p-3)}{p(4p^2-11p+8)}$, then Theorem 3.2 holds.

Proof idea of the lower bound of Theorem 3.2

The next lemma says that for any $R^* \in \mathcal{R}$ satisfying the conclusion of Lemma 4.2, one may assume that the set $A_i(R^*)$ for every $i \in [p]$ is non-empty.

Proof idea of the lower bound of Theorem 3.2

The next lemma says that for any $R^* \in \mathcal{R}$ satisfying the conclusion of Lemma 4.2, one may assume that the set $A_i(R^*)$ for every $i \in [p]$ is non-empty.

Lemma 4.4

Suppose that $\mathcal{R}$ is under the condition (i). Let $R^* \in \mathcal{R}$ be any clique satisfying (3). If there exists some $i \in [p]$ such that $A_i(R^*) = \emptyset$, then we can get that

$$\ell_2^{\mathcal{R}} \geq \frac{(2(p-2) - p(2p-3)r)^2}{8(p-1)^3} n^2 - \frac{2(p-2) - p(2p-3)r}{4(p-1)} n,$$

and Theorem 3.2 holds.

Proof of the key lemma

Finally we are ready to finish the proof of Theorem 3.2. By Lemma 4.4, for any $\mathcal{R}$ satisfying the condition (i) and for any $R_0 \in \mathcal{R}$ satisfying (3), we may assume that $A_i(R_0) \neq \emptyset$ for each $i \in [p]$, i.e., any $p-1$ vertices in $V(R_0)$ have at least one common neighbor in $H_{\mathcal{R}} = G \backslash V(\mathcal{R})$.

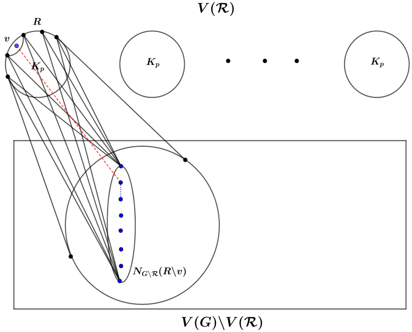

Figure 10. The structure between R and $N_{H_{\mathcal{R}}}(R)$.

Proof idea of the lower bound of Theorem 3.2

- Let $R^* \in \mathcal{R}$ be the p-clique obtained from Lemma 4.2. So R^* satisfies (3).

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- Since $A_i(R^*) \neq \emptyset$ for each $i \in [p]$, such a clique C exists in $H_{\mathcal{R}}$ (for instance, one can just take one vertex in $A_1(R^*)$).
- Let $V(R^*) = \{v_1, ..., v_p\}$ and $V(C) = \{x_1, ..., x_c\}$ for some integer $c \geq 1$. Without loss of generality we may assume that

$$V(R') = \{x_1, ..., x_c, v_{c+1}, ..., v_p\}.$$

Proof idea of the lower bound of Theorem 3.2

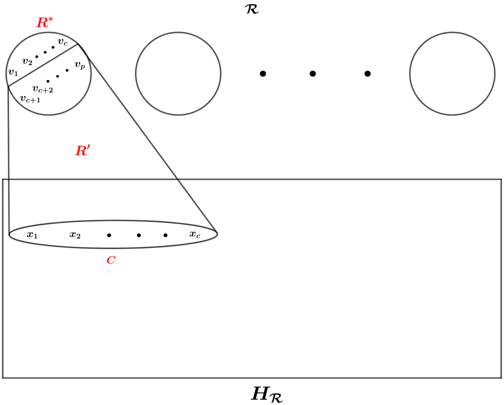

Figure 11. The structure of R^* and R'.

Proof idea of the lower bound of Theorem 3.2

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Proof idea of the lower bound of Theorem 3.2

- In what follows, we should complete the proof by deriving the final contradiction that $c \geq p$.
- Suppose that $c \leq p-1$. In this case, we are always able to find a clique in $H_{\mathcal{R}}$ of larger size than C and satisfying the above conditions required for C.
- To see this, let $\mathcal{R}' = (\mathcal{R} \backslash \{R^*\}) \cup \{R'\}$ and $H_{\mathcal{R}'} = G \backslash V(\mathcal{R}')$.
- So $\mathcal{R}'$ also satisfies the condition (i) and

$$V(H_{\mathcal{R}'}) = (V(H_{\mathcal{R}}) \backslash \{x_1, \ldots, x_c\}) \cup \{v_1, \ldots, v_c\}.$$

Proof idea of the lower bound of Theorem 3.2

- Applying Lemma 4.1 with the clique R therein being R^*, we know that

$$e(R', H_{\mathcal{R}'}) \geq e(R^*, H_{\mathcal{R}}) \geq \left(\frac{p(p-2)}{p-1} - \frac{p(2p^2-4p+1)}{2(p-1)} r \right) n,$$

where the last inequality holds as R^* satisfies (3). That says, $R' \in \mathcal{R}'$ also satisfies (3).

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- As discussed earlier, by Lemma 4.4, any $p-1$ vertices in $V(R')$ have at least one common neighbor in $H_{\mathcal{R}'}$.

- In particular, there exists a vertex $y \in V(H_{\mathcal{R}'})$ such that it is not adjacent to v_p but is adjacent to all other vertices of $V(R')$.

Proof idea of the lower bound of Theorem 3.2

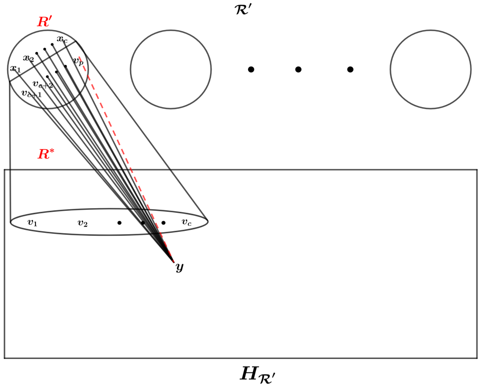

Figure 12. Find a larger clique than C satisfying the condition.

Proof idea of the lower bound of Theorem 3.2

- Obviously, $y \notin \{v_1, \ldots, v_c\}$, since $v_i v_p \in E(G)$ for each $i \in [c]$. So it must be the case that $y \in V(H_{\mathcal{R}}) \setminus \{x_1, \ldots, x_c\}$.

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- Then C' is a clique in $H_{\mathcal{R}}$ of size larger than C such that $C' \cup \{v_{c+1}, \ldots, v_{p-1}\}$ is a p-clique contained in $R^* \cup C'$ and covering all vertices of C'.

Proof idea of the lower bound of Theorem 3.2

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- Now let $C' = \{x_1, \ldots, x_c, y\} \subseteq V(H_{\mathcal{R}})$.
- Then C' is a clique in $H_{\mathcal{R}}$ of size larger than C such that $C' \cup \{v_{c+1}, \ldots, v_{p-1}\}$ is a p-clique contained in $R^* \cup C'$ and covering all vertices of C'.
- This is a contradiction to our choice of C. Therefore, we must have that $c \geq p$.

Proof idea of the lower bound of Theorem 3.2

- Obviously, $y \notin \{v_1, \ldots, v_c\}$, since $v_i v_p \in E(G)$ for each $i \in [c]$. So it must be the case that $y \in V(H_{\mathcal{R}}) \setminus \{x_1, \ldots, x_c\}$.
- Now let $C' = \{x_1, \ldots, x_c, y\} \subseteq V(H_{\mathcal{R}})$.
- Then C' is a clique in $H_{\mathcal{R}}$ of size larger than C such that $C' \cup \{v_{c+1}, \ldots, v_{p-1}\}$ is a p-clique contained in $R^* \cup C'$ and covering all vertices of C'.
- This is a contradiction to our choice of C. Therefore, we must have that $c \geq p$.
- However, it is also a contradiction to the fact that $H_{\mathcal{R}}$ is K_p-free, which completes the proof of Theorem 3.2.

Thanks for your attention!