# An Introduction to DP color functions

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## Outline





## Notations

- $G = (V(G), E(G))$.
- $\mathbb{N}$: the set of positive integers.
- $\forall m \in \mathbb{N}$, let $[m] := \{1, 2, \ldots, m\}$.
- Note: in this talk, $m$ doesn't represent the number of edges.



## Proper coloring

- A **proper coloring**: a mapping $c : V(G) \to \mathbb{N}$, such that $c(u) \neq c(v)$ for all $uv \in E(G)$.
- A **proper $m$-coloring**: a proper coloring $c$ with $c(u) \in [m]$ for all $u \in V(G)$.


Figure: Two proper 2-colorings of $C_4$

- The **chromatic polynomial** $P(G, m)$: the number of proper $m$-colorings, for each $m \in \mathbb{N}$.

# List coloring

- Introduced by Vizing and Erdős, Rubin, Taylor independently.
- An **$m$-list assignment $L$**: a mapping $L$ from $V(G)$ to $2^{\mathbb{N}}$, such that $|L(v)| = m$ holds for all $v \in V(G)$.
- Examples:

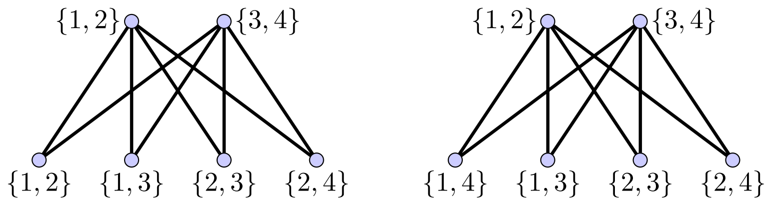



Figure: 2-list assignments of $K_{2,4}$

- $L(v) := [m]$ for all $v \in V(G)$.



# List coloring

- An **$L$-coloring**: a proper coloring $c$ with $c(v) \in L(v)$ for all $v \in V(G)$.
- Examples:

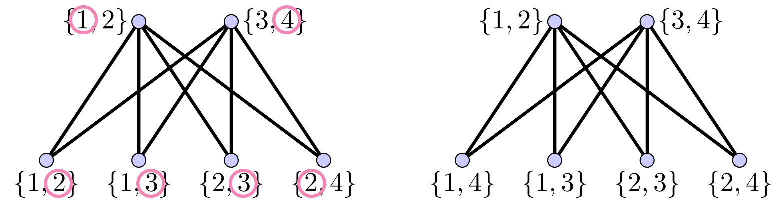



Figure: $K_{2,4}$ with a 2-list assignment $L$

- for the $L$ with $L(v) = [m]$ for all $v \in V(G)$, each proper $m$-coloring is an $L$-coloring.



# List coloring

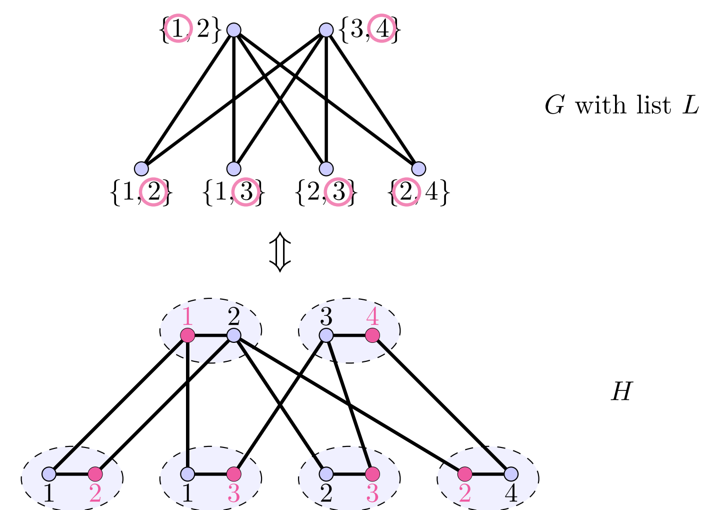
- $P(G, L)$: the number of $L$-colorings.
- The **list color function** $P_l(G, m)$: the minimum value of $P(G, L)$ among all $m$-list assignments $L$, for each $m \in \mathbb{N}$.
- By definition,

$$P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}. \tag{1}$$



# From list coloring to DP coloring

- Introduced by Dvořák and Postle in 2018.



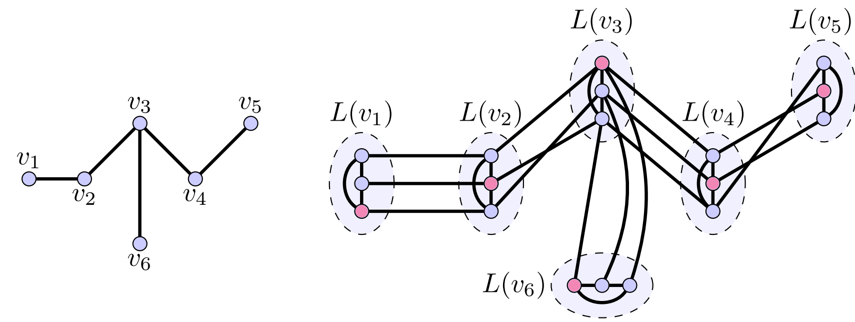
## DP coloring

- $\color{red}{E_G(U,V)} := \{uv \in E(G) : u \in U, v \in V\}$.
- An ***m*-fold cover**: an ordered pair $\mathcal{H} = (L, H)$, where $H$ is a graph and $L$ is a mapping from $V(G)$ to $2^{V(H)}$ satisfying the conditions below:
  - for every $u \in V(G)$, $L(u)$ is an $m$-clique in $H$,
  - the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$,
  - if $uv \notin E(G)$, then $E_H(L(u), L(v)) = \emptyset$, and
  - if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.



## DP coloring

$T$ with a 3-fold cover $\mathcal{H} = (L, H)$.



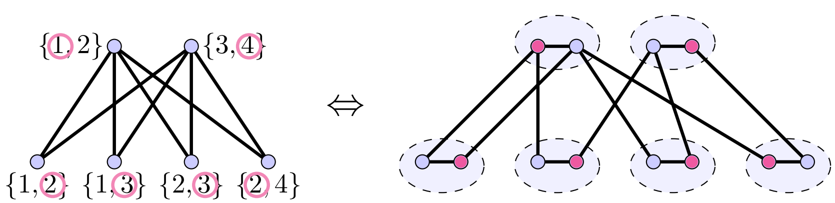

(a) $T$ (b) $H$

- an **$\mathcal{H}$-coloring**: an independent set in $H$ with size $|V(G)|$.



## DP coloring

- For every $m$-list assignment $\color{red}{L}$, there is a corresponding $m$-fold cover $\color{red}{\mathcal{H} = (L', H)}$:
  - $V(H) = \bigcup_{u \in V(G)} L'(u)$,
  - $L'(u) = \{(u,i) : i \in L(u)\}$ for every $u \in V(G)$, and
  - $(u,i)(v,j) \in E(H)$ iff $u = v$, or $uv \in E(G)$ and $i = j$.

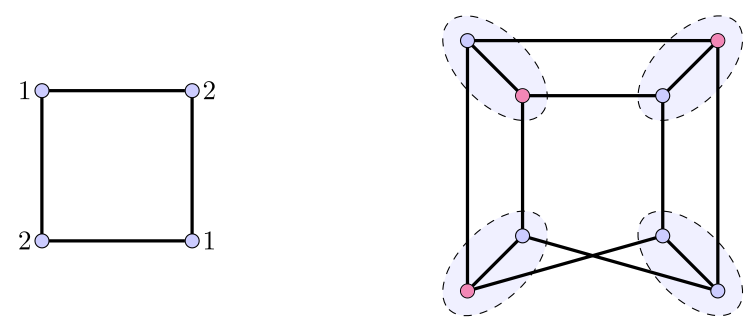



(a) an $L$-coloring (b) an $\mathcal{H}$-coloring



## DP coloring

$C_4$ with a 2-fold cover $\mathcal{H} = (L, H)$.




(a) $C_4$ (b) $\mathcal{H}$

## DP coloring

- $P(G, \mathcal{H})$: the number of $\mathcal{H}$-colorings.
- The **DP color function** $P_{DP}(G, m)$: the minimum value of $P(G, \mathcal{H})$ among all $m$-fold covers $\mathcal{H}$, for each $m \in \mathbb{N}$.
- By definition,

$$P_{DP}(G, m) \leq P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}. \tag{2}$$

- Note that all the equalities in (2) hold when $G$ is a chordal graph.



## Three color functions

$$P_{DP}(G, m) \leq P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}.$$

*Donner, 1992*

$P_l(G, m) = P(G, m)$ holds when $m$ is sufficiently large.

*Thomassen, 2009*

$P_l(G, m) = P(G, m)$ holds when $m > |V(G)|^{10}$.

*Wang, Qian and Yan, 2017*

$P_l(G, m) = P(G, m)$ holds when $m > \frac{|E(G)|}{\log(1+\sqrt{2})} \approx 1.135(|E(G)| - 1)$.



## Three color functions

$$P_{DP}(G, m) \leq P_l(G, m) \leq P(G, m), \ \forall\, m \in \mathbb{N}.$$

*F.M. Dong and M.Q. Zhang, 2023*

$P_l(G, m) = P(G, m)$ holds whenever $m \geq |E(G)| - 1$.

- However, the DP color functions of some graphs might not tend to be the same as their chromatic polynomials.

*Kaul and Mudrock, 2019*

If the girth of $G$ is even, then there exists $M \in \mathbb{N}$, such that $P_{DP}(G, m) < P(G, m)$ for all integers $m \geq M$.



## Between $P_{DP}(G, m)$ and $P(G, m)$

- Therefore, two sets of graphs $\mathbf{DP}_<$ and $\mathbf{DP}_\approx$ are naturally defined.
  - $DP_<$: the set of graphs $G$ for which there is $M \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ holds for all integers $m \geq M$, and
  - $DP_\approx$: the set of graphs $G$ for which there is $M \in \mathbb{N}$ such that $P_{DP}(G, m) = P(G, m)$ holds for all integers $m \geq M$.
- Note: a characterization of the graphs in set $DP_<$ or $DP_\approx$ does not necessarily guarantee a characterization of the graphs in the other set.

## Known results on $DP_<$

- For any $e \in E(G)$, let $\mathcal{C}_G(e)$ be the set of shortest cycles in $G$ containing $e$.
- The **girth of edge** $e$, denoted by $\ell_G(e)$:
  - $\infty$ if $\mathcal{C}_G(e) = \emptyset$;
  - the size of any cycle in $\mathcal{C}_G(e)$ otherwise.

*Dong and Yang, 2022*

Graph $G$ belongs to $DP_<$ if $G$ contains an edge whose girth is even.

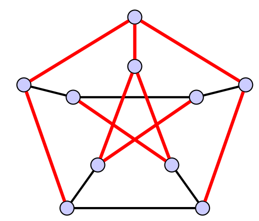


## Known results on $DP_\approx$

- On the other hand, Mudrock and Thomason first showed that each graph with a dominating vertex belongs to $DP_\approx$.

*Dong and Yang, 2022*

Graph $G$ belongs to $DP_\approx$ if $G$ has a spanning tree $T$ such that for each edge $e$ in $E(G) \setminus E(T)$, $\ell_G(e)$ is odd and there exists a cycle $C \in \mathcal{C}_G(e)$, where $\ell_G(e') < \ell_G(e)$ holds for each $e' \in E(C) \setminus (E(T) \cup \{e\})$.

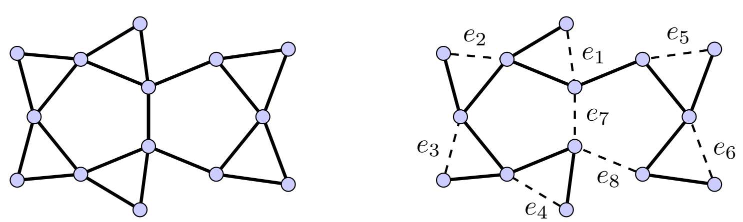




## Our results on $DP_\approx$

*M.Q. Zhang and F.M. Dong, 2023*

Graph $G$ belongs to $DP_\approx$ if $G$ has a spanning tree $T$ and a labeling $e_1, \ldots, e_q$ of all the edges in $E(G) \setminus E(T)$, such that $\ell_G(e_1) \le \cdots \le \ell_G(e_q)$ and for each $i \in [q]$, $\ell_G(e_i)$ is odd and $E(C_{e_i}) \subseteq E(T) \cup \{e_1, \ldots, e_i\}$ holds for some $C_{e_i} \in \mathcal{C}_G(e_i)$.




(a) $G$ (b) $T$ and an edge labeling



## Our results on $DP_\approx$

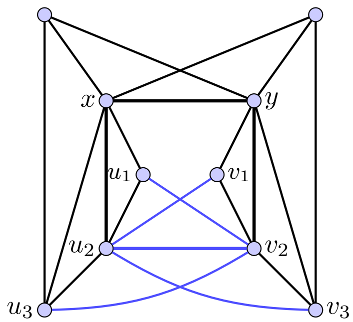
*M.Q. Zhang and F.M. Dong, 2023*

Let $G$ be a graph with vertex set $\{v_i : i = 0, 1, \ldots, n\}$, where $n \ge 1$. If for each $i \in [n]$, the set $N(v_i) \cap \{v_j : 0 \le j \le i-1\}$ is not empty and the subgraph of $G$ induced by this vertex set is connected, then $G$ is in $DP_\approx$.

- Immediately, many special classes of graphs belong to $DP_\approx$, such as complete $k$-partite graphs with $k \ge 3$ and plane near-triangulations.

# Our results on $DP_<$

- For any $E^* \subseteq E(G)$, let $\mathcal{C}_G(E^*)$ be the set of shortest cycles $C$ in $G$ such that $|E(C) \cap E^*|$ is odd.
- The **girth of edge set** $E^*$, denoted by $\ell_G(E^*)$:
  - $\infty$ if $\mathcal{C}_G(E^*) = \emptyset$;
  - the size of any cycle in $\mathcal{C}_G(E^*)$ otherwise.



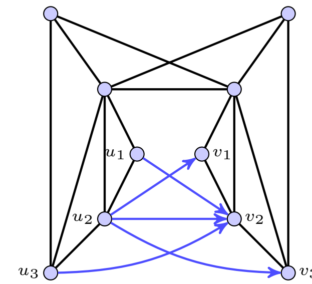

$E^* = \{u_1v_2, u_2v_1, u_2v_2, u_2v_3, u_3v_2\}$

$\ell_G(E^*) = 4$



# Our results on $DP_<$

- For any $E^* \subseteq E(G)$, assume that each $e$ in $E^*$ is assigned a direction $\overrightarrow{e}$ and only edges in $E^*$ are assigned directions.



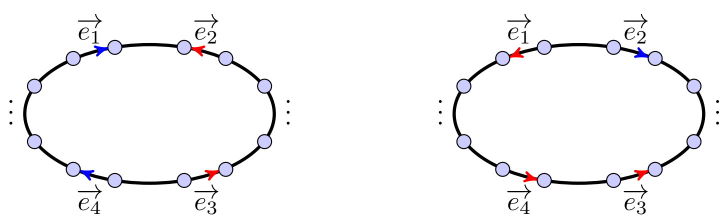

- Let $\overrightarrow{E^*}$ be the set of directed edges $\overrightarrow{e}$ for all $e \in E^*$.

$\Rightarrow \overrightarrow{E^*} = \{\overrightarrow{u_1v_2}, \overrightarrow{u_2v_1}, \overrightarrow{u_2v_2}, \overrightarrow{u_2v_3}, \overrightarrow{u_3v_2}\}$.



# Our results on $DP_<$

- For any cycle $C$, we say the directed edges in $\overrightarrow{E^*}$ are **balanced on $C$** if $|E(C) \cap E^*|$ is even, and exactly half of the edges in $E(C) \cap E^*$ are oriented clockwise along $C$.




(a) Balanced (b) Unbalanced

Figure: $\overrightarrow{E^*} = \{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}\}$



# Our results on $DP_<$

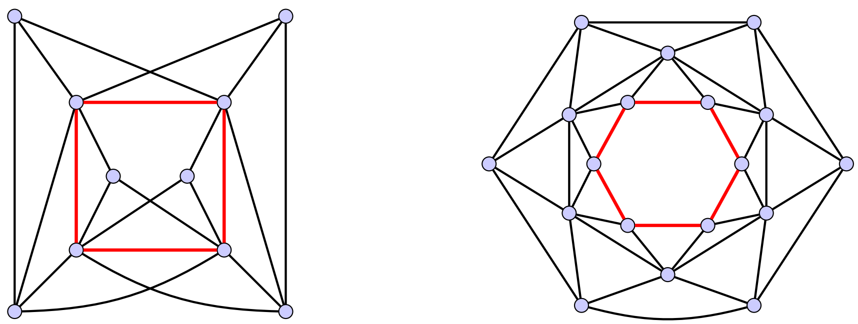
*M.Q. Zhang and F.M. Dong, 2023*

Let $G$ be a connected graph and $E^*$ be a set of edges in $G$. Assume that

1. $\ell_G(E^*)$ is even; and
2. there exists a way to assign a direction $\overrightarrow{e}$ for each edge $e \in E^*$ such that the directed edges in $\overrightarrow{E^*} = \{\overrightarrow{e} : e \in E^*\}$ are balanced on each cycle $C$ of $G$ with $|E(C)| < \ell_G(E^*)$.

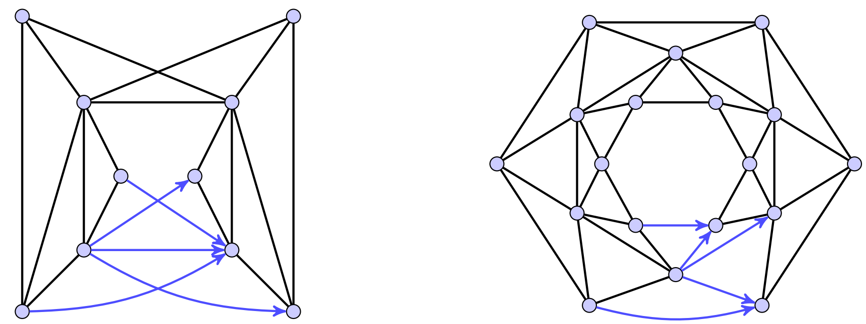
Then $P(G,m) - P_{DP}(G,m) \geq \Omega(m^{|V(G)|-\ell_G(E^*)+1})$ holds, and hence $G \in DP_<$.

## Our results on $DP_<$





## Our results on $DP_<$



$\ell_G(E^*) = 4$    $\ell_G(E^*) = 6$

*M.Q. Zhang and F.M. Dong, 2023*

Let $G$ be any graph and let $E^* \subseteq E_G(V_1, V_2)$, where $V_1$ and $V_2$ are disjoint vertex subsets of $V(G)$. If $\ell_G(E^*) = 4$, then $P(G,m) - P_{DP}(G,m) \geq \Omega(m^{n-3})$ holds, and hence $G \in DP_<$.



## Future Research

**Question**

How to characterize the graphs in sets $DP_<$ and $DP_\approx$?

**Question**

What is the property of an $m$-fold cover $\mathcal{H}$ of $G$ with $P(G,\mathcal{H}) = P_{DP}(G,m)$?

**Question**

Is there any graph not in $DP_< \cup DP_\approx$?



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