On Circular Chromatic Number of Signed Planar Graphs of Girth At Least 5

Zhiqian Wang

Join works with Jiaao Li

Nankai Unversity

December 9th, 2022

Circular coloring of a graph

The concept of circular coloring of a graph was introduced by Vince in 1988. Let C^r be the circle of circumference r obtained from $[0, r]$ by identifying 0 and r.

Definition 1

A circular coloring of graph G is a mapping $f : V(G) \mapsto C^r$ such that for any edge $xy \in E(G)$, $d_{(\mathrm{mod}\, r)}(f(x), f(y)) \geq 1$. The circular chromatic number of G is

$$\chi_c(G) = \inf\{r\text{: G admits a circular r-coloring}\}.$$

Circular coloring of a signed graph

In 2021, R. Naserasr, Z. Wang and X. Zhu extended the concept of circular coloring to signed graphs.

Definition 2

A signed graph is a graph G together with an assignment $\sigma : E(G) \mapsto \{+, -\}$, denoted by (G, σ).

Circular coloring of a signed graph

Given a point x on C^r, the antipodal of x is denoted by $\bar{x}$.

Definition 3

Given a signed graph (G, σ) and a real number r, a circular coloring of graph G is a mapping $f : V(G) \mapsto C^r$ such that for each positive edge $xy \in E(G)$, $d_{(\mathrm{mod}\, r)}(f(x), f(y)) \geq 1$ and for each negative edge $xy \in E(G)$, $d_{(\mathrm{mod}\, r)}(f(x), \overline{f(y)}) \geq 1$. In other words,

- $1 \leq |f(x) - f(y)| \leq r - 1$ if xy is positive;
- $|f(x) - f(y)| \leq \frac{r}{2} - 1$ or $|f(x) - f(y)| \geq \frac{r}{2} + 1$ if xy is negative.

The circular chromatic number of (G, σ) is defined similarly.

Circular coloring of a signed graph

Specially, if r is a rational number $\frac{p}{q}$ (p is even), the discrete version of a circular coloring is that

Definition 4

Given an even integer p and an integer q,where $q \leq \frac{p}{2}$, a (p, q)-coloring is a mapping $f : V(G) \mapsto C^r$ such that:

- For each positive edge e, $|f(x) - f(y)| \in \{q, \ldots, p - q\}$.
- For each negative edge e, $|f(x) - f(y)| \in \{0, \ldots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \ldots, p - 1\}$.

Circular coloring of a signed graph

Theorem 5

$\chi_c(G) = 2$ if G is a forest; otherwise G has a cycle with s positive edges and t negative edges such that $\chi_c(G) = \frac{2(s+t)}{2a+t}$ for some integer a.

And they made some preliminary estimation for $\chi_c(G)$ of a planar graph G.

Theorem 6

$\chi_c(G) \leq \frac{10}{3}$ for a signed outerplanar graph G.

Theorem 7

$\chi_c(G) \leq 4$ for a signed bipartite graph G, and there exists a sequence of signed bipartite planar graphs with χ_c tending to 4.

R. Naserasr, Z. Wang, X. Zhu, Circular Chromatic Number of Signed Graphs, The Electronic Journal of Combinatorics, Volume 28, Issue

Circular coloring of a signed graph

They strengthened the bound for signed bipartite planar graphs in their subsequent work:

Theorem 8

If (G, σ) is a signed bipartite planar simple graph on n vertices, then $\chi_c(G) \leq 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$, and the bound is tight for each n.

F. Kardos, J. Narboni, R. Naserasr, Z. Wang, Circular $(4-\varepsilon)$-coloring of some classes of signed graphs, SIAM Journal on Discrete Mathematics, 2022+.

Circular flow of a signed graph

As a dual concept of circular coloring, circular flow was introduced by L. A. Goddyn, M. Tarsi and C. Q. Zhang in 1998. recently, J. Li et has extend it to signed graphs.

Definition 9

Given a signed graph (G, σ) and a real number $\mathrm{r} \geq 2$, a circular r-flow is a pair (D, f) where D is an orientation and $\mathrm{f} : \mathrm{E}(\mathrm{G}) \mapsto (-\mathrm{r}, \mathrm{r})$ satisfies the following conditions:

- For each positive edge e, $|\mathrm{f}(\mathrm{e})| \in [1, \mathrm{r}-1]$.
- For each negative edge e, $|\mathrm{f}(\mathrm{e})| \in [0, \frac{\mathrm{r}}{2}-1] \cup [\frac{\mathrm{r}}{2}+1, \mathrm{r})$.
- For each vertex v, $\partial(\mathrm{f}(\mathrm{v})) = \sum_{\mathrm{e} \in \mathrm{E}^+(\mathrm{v})} \mathrm{f}(\mathrm{e}) - \sum_{\mathrm{e} \in \mathrm{E}^-(\mathrm{v})} \mathrm{f}(\mathrm{e}) = 0$.

Circular flow of a signed graph

Definition 10

The circular flow index $\Phi_c((G,\sigma))$ is defined as

$$\Phi_c((G,\sigma)) = \inf\{\text{r: G admits a circular r-flow}\}.$$

J. Li, R. Naserasr, Z. Wang, X. Zhu, Circular flows in mono-directed signed graphs, Journal of Combinatorial Theory, Series B, 2022+.

Circular flow of a signed graph

Definition 11

Given a signed graph (G, σ) and a real number $r \geq 2$, a circular modulo r-flow is a pair (D, f) where D is an orientation and $f : E(G) \mapsto [0, r)$ satisfies the following conditions:

- For each positive edge e, $|f(e)| \in [1, r-1]$.
- For each negative edge e, $|f(e)| \in [0, \frac{r}{2} - 1] \cup [\frac{r}{2} + 1, r)$.
- For each vertex v, $\partial(f(v)) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0 \pmod{r}$.

Circular flow of a signed graph

Specially, if r is a rational number $\frac{p}{q}$, the discrete version of a circular (modulo) r-flow is that

Definition 12

Given an even integer p and an integer q,where $q \leq \frac{p}{2}$, a (modulo) (p, q) is a pair (D, f) where D is an orientation and $f : E(G) \mapsto \mathbb{Z}_p$ satisfies the following conditions:

- For each positive edge e, $|f(e)| \in \{q, \ldots, p - q\}$.
- For each negative edge e, $|f(e)| \in \{0, \ldots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \ldots, p - 1\}$.
- For each vertex v, $\partial(f(v)) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0 \pmod{p}$.

Circular flow of a signed graph

The same as in unsigned graphs, flow and modulo flow are equivalent:

Theorem 13

(G, σ) admits a circular r-flow if and only if it admits a circular modulo r-flow.

J. Li, R. Naserasr, Z. Wang, X. Zhu, Circular flows in mono-directed signed graphs, Journal of Combinatorial Theory, Series B, 2022+.

Circular flow of a signed graph

Already known results about edge connectivity and circular flow index:

Edge connectivity	Φ_c
$6p-2$	$\leq \frac{8p-2}{4p-3}$
$6p-1$	$\leq \frac{4p}{2p-1}$
$6p$	$< \frac{4p}{2p-1}$
$6p+1$	$\leq \frac{8p+2}{4p-1}$
$6p+2$	$\leq \frac{2p+1}{p}$
$6p+3$	$< \frac{2p+1}{p}$

J. Li, R. Naserasr, Z. Wang, X. Zhu, Circular flows in mono-directed signed graphs, Journal of Combinatorial Theory, Series B, 2022+.

Circular flow of a signed graph

Theorem 14

$\Phi_c((G,\sigma)) \leq 4$ if (G,σ) is 4-edge-connected.

Theorem 15

$\Phi_c((G,\sigma)) < 4$ if (G,σ) is 6-edge-connected.

Especially there are better bounds for signed Eulerian graphs:

Theorem 16

For any signed Eulerian graph (G,σ), if G is $(6p-2)$-edge-connected, then $\Phi_c((G,\sigma)) \leq \frac{4p}{2p-1}$. Equivalently, every signed bipartite planar graph of girth at least $6p-2$ admits a circular $\frac{4p}{2p-1}$-coloring.

J. Li, R. Naserasr, Z. Wang, X. Zhu, Circular flows in mono-directed signed graphs, Journal of Combinatorial Theory, Series B, 2022+.

Circular flow of a signed graph

And the result has been strengthened recently:

Theorem 17

For each $p \in \{1, 2, 3, 4\}$ and every $(6p-4)$-edge-connected signed Eulerian graph (G, σ), $\Phi_c((G, \sigma)) \leq \frac{4p}{2p-1}$. Equivalently, every signed bipartite planar graph of negative girth at least $6p-4$ admits a circular $\frac{4p}{2p-1}$-coloring.

J. Li, Y. Shi, Z. Wang, C. Wei, Strongly $\mathbb{Z}_{2p}$-connectedness and mapping signed bipartite planar graphs to cycles, 2022+.

Circular flow of a signed graph

Conjecture 18

(3-flow Conjecture) Every 4(or 5)-edge-connected graph admits a 3-flow.

Theorem 19

(C. Thomassen et, 2016) Every 5-edge-connected planar graph is $\mathbb{Z}_3$-connected.

R. B. Richter, C. Thomassen, D. Younger, Group-colouring, group-connectivity, claw-decompositions, and orientations in 5-edge-connected planar graphs, Journal of Combinatorics, Volume 7, 219–232, 2016.

5-edge-connected planar graphs

We studied the flow index of 5-edge-connected signed planar graphs and obtained the following result:

Theorem 20

For every 5-edge-connected signed planar graph (G, σ) of m edges, $\Phi_c((G, \sigma)) \leq 4 - \frac{1}{m+2}$. Equivalently, for every signed planar graph of girth at least 5, $\chi_c(G) \leq 4 - \frac{1}{m+2} \leq 4 - \frac{12}{5n-4}$.

5-edge-connected planar graphs

Actually we worked on a more general case rather than 4-flow itself. Say a mapping $\beta : V(G) \mapsto \{0, 1, 2, 3\}$ is called a $\mathbb{Z}_4$-boundary of G if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 4$. Furthermore, we define the $\mathbb{Z}_4$-boundary of any subset $A \in V(G)$ by $\beta(A) = \sum_{v \in A} \beta(v) \pmod 4$. A 2-cut $[A, A^c]$ is said to be bad if $\beta(A) = \beta(A^c) \equiv 2 \pmod 4$.

5-edge-connected planar graphs

Theorem 21

G is a 3-edge-connected planar graph embedded in the plane. Given any $\mathbb{Z}_4$-boundary β, if G has at most two specified vertices d and t such that:
(i) if d exists, then it has degree 3, 4, or 5, has its incident edges oriented and labelled with $\{1, 2\} \in \mathbb{Z}_4$, and is in the boundary of the unbounded face;
(ii) if t exists, then it has degree 3 and is in the boundary of the unbounded face;
(iii) there are at most two 3-cuts, which can only be $\partial(\mathrm{d})$ and $\partial(\mathrm{t})$;
(iv) if d has degree 5, then t does not exist;
(v) every vertex not in the boundary of the unbounded face has five edge disjoint paths to the boundary of the unbounded face;
(vi) $\mathrm{G} - \mathrm{d}$ has no bad 2-cut.

5-edge-connected planar graphs

Theorem 22

Then for every $\mathbb{Z}_4$-boundary β, if the prescription at d is $\beta(\mathrm{d})$, it can be extended to a flow f with an orientation D such that:
(a) the boundary of f is β, that is, $\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) \equiv \beta(v) \pmod 4$ for every $v \in V(G)$;
(b) $f(e) \in \{1, 2\}$ for every $e \in E(G)$;
(c) D is strong on $G - d$.
If d does not exist, we take $d = \emptyset$ and $G - d = G$.

5-edge-connected planar graphs I

Theorem 21 $\Rightarrow$ Theorem 20: Define $\mathrm{d}^-(\mathrm{v})$ to be the number of negative edges v is incident to, and a $\mathbb{Z}_4$-boundary $\tilde{\beta}(\mathrm{v}) \equiv 2 \cdot \mathrm{d}^-(\mathrm{v}) \pmod 4$. By Theorem 21, the unsigned underlying graph G_0 of G admits $\tilde{\mathrm{f}}$: $\mathrm{E}(\mathrm{G}_0) \mapsto \{1, 2\}$ with boundary $\tilde{\beta}$ with a strong orientation D of G.
Define f': $\mathrm{E}(\mathrm{G}) \mapsto \mathbb{Z}_4$:

$$\mathrm{f}'(\mathrm{e}) = \begin{cases} 2 \pmod 4 & \text{e is negative;} \\ 0 \pmod 4 & \text{e is positive.} \end{cases}$$

For $\mathrm{v} \in \mathrm{V}(\mathrm{G})$, $\mathrm{f}^+(\mathrm{v}) - \mathrm{f}^-(\mathrm{v}) \equiv 2 \cdot \mathrm{d}^-(\mathrm{v}) \pmod 4$. Then $\hat{\mathrm{f}} = \tilde{\mathrm{f}} + \mathrm{f}'$ is a $\mathbb{Z}_4$-flow of G, with $\hat{\mathrm{f}}(\mathrm{e}) \in \{1, 2\}$ if e is positive and $\hat{\mathrm{f}}(\mathrm{e}) \in \{0, 3\}$ if e is negative.

5-edge-connected planar graphs II

Since D is strong, every edge ε in G is contained in an oriented cycle C_ε. For $\varepsilon \in \mathrm{E(G)}$, define f_ε: $\mathrm{E(G)} \mapsto \{0,1\}$:

$$\mathrm{f}_\varepsilon(\mathrm{e}) = \begin{cases} 1 & \mathrm{e} \in \mathrm{C}_\varepsilon; \\ 0 & \text{otherwise.} \end{cases}$$

Write $\mathrm{m}' = \mathrm{m}+1$, $\mathrm{f} = \mathrm{m}'\hat{\mathrm{f}} + \sum_{\varepsilon \in \mathrm{E(G)}} \mathrm{f}_\varepsilon$ is a $\mathbb{Z}_{4\mathrm{m}'}$-flow of G, with $\mathrm{f(e)} \in \{\mathrm{m}'+1, \ldots, 3\mathrm{m}'-1\}$ if e is positive and $\mathrm{f(e)} \in \{1, \ldots, \mathrm{m}'-1\} \bigcup \{3\mathrm{m}'+1, \ldots, 4\mathrm{m}'-1\}$ if e is negative. So f is a $\dfrac{4\mathrm{m}'}{\mathrm{m}'+1} \pmod{4\mathrm{m}'}$-flow of G, and $\phi(\mathrm{G}) \leqslant \dfrac{4\mathrm{m}'}{\mathrm{m}'+1} < 4$.

Assume $\ddot{G}$ is the minimal counterexample of Theorem 21. We study the necessary properties of $\ddot{G}$ and list them by a series of propositions.

Claim 23

$\ddot{G}$ does not contain unoriented multiedges.

Claim 24

$\ddot{G}$ is 2-connected.

5-edge-connected planar graphs

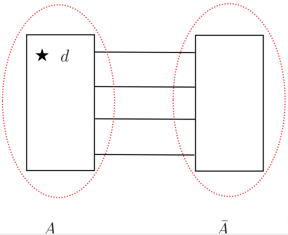
We say a cut $[A, A^c]$ of G is essential if both $|A|, |A^c| \geq 2$.

Claim 25

There does not exist an essential 4-cut $[A, A^c]$ such that $d \in A$ and $\ddot{G}[A^c]$ is 3-edge-connected.

Claim 26

There does not exist an essential 5-cut $[A, A^c]$ such that $\{d, t\} \subseteq A$, and $\ddot{G}[A^c]$ is 3-edge-connected.



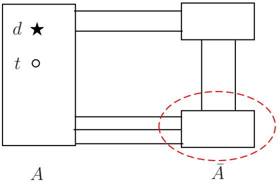
5-edge-connected planar graphs

Claim 27

There is no essential 4-cut in $\ddot{G}$.

Claim 28

There does not exist an essential 5-cut $[A, A^c]$ such that $\{d, t\} \subseteq A$.



5-edge-connected planar graphs

Claim 29

d exists in $\ddot{G}$, and $\deg(d) = 3$ or 4.

Claim 30

t exists, and d and t are non-adjacent.

Proof: If the boundary vertices (prescribed or not) are saturated enough, we can orient and label an edge e (e may also has been already prescribed if it is incident to d) on the boundary and apply induction on $G - e$. Then d and t both exist, and $\deg(d) \neq 5$. If d and t are adjacent, $\partial(\{d, t\})$ is a cut violating Claim 27 or Claim 28.

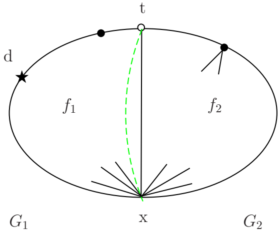
Corollary 31

$G - d$ is 3-edge-connected.

5-edge-connected planar graphs

Claim 32

t is not incident to a chord.



Claim 33

There are at most two 2-cuts in $G - d - t$, and t can be prescribed properly without leaving any bad 2-cut.

5-edge-connected planar graphs

A 2-cut in $G - d - t$ corresponds to a 3-cut in $G - d$, such a cut is formed by a neighbour of t, and there are at most three.

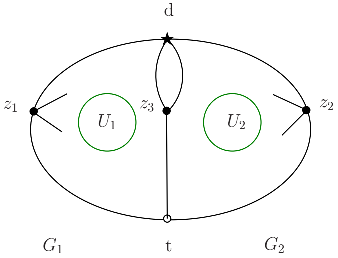


Figure: The configuration of coexisting three cuts

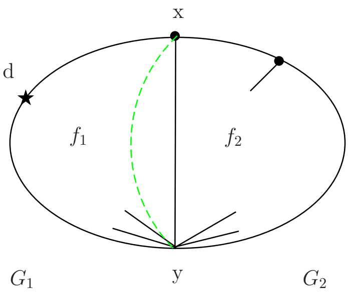
Corollary 34

The degree of boundary neighbours of t is exactly 4.

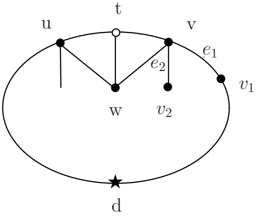
5-edge-connected planar graphs

Claim 35

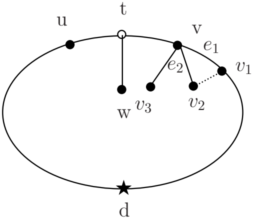
A 4-vertex is not incident to an unoriented chord.



Deduced from the claims before, the two possible configurations of $\ddot{G}$ are:



(a) Case I



(b) Case II

Deal with the two cases respectively.

5-edge-connected planar graphs

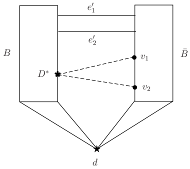
For Case I, orient and label e_1 and e_2 to realize $\beta(v)$, contract the diamond $\{u, t, v, w\}$ and apply induction on the resulting graph G'. A strong flow D of $G' - d$ can be extended to a strong flow of $G - d$:
This can be done by prescribing u and t wisely within the diamond D^*. First, uw can be oriented so that D is strongly extended on $G - d - t - v$, and we prescribe u with this restriction to achieve $\beta(u)$. The value and orientation of ut may be locked but still, we can prescribe t properly and since the boundary value of v in D^* is 0, we prescribe vw with the same value and opposite orientation of tv. As a result D is extended to a strong flow of $G - d - \{e_1, e_2\}$, and thus of $G - d$.

5-edge-connected planar graphs

Lemma 36

There is at most one possible 2-cut in $G' - d$, which can only be $\partial(v_1) \cap E(G' - d)$.

The BAD CASE:



Contract $B \cup \{d\}$ to a vertex d_1; contract $\bar{B} \cup \{d\}$ to a vertex d_2.

5-edge-connected planar graphs

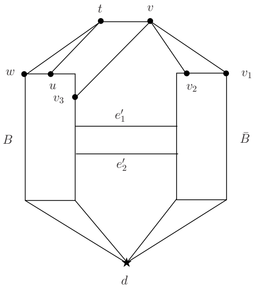
For Case II, lift e_1 and e_2 at v (that is, delete e_1 and e_2, then add a new edge v_1v_2), orient and label the remaining two edges to realize $\beta(v)$, and properly prescribe t with the other two edges to realize $\beta(t)$. Then delete $\{v, t\}$ and apply induction on the resulting graph G''. A strong flow D of $G'' - d$ can be extended to a strong flow of $G - d$ because t is prescribed to be neither a sink nor a source.

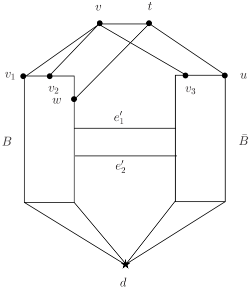
5-edge-connected planar graphs

Lemma 37

There is at most one 2-cut in $G'' - d$, which can only be $\partial(u) \cap E(G'' - d)$.

Two BAD CASES:

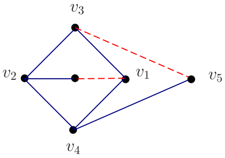




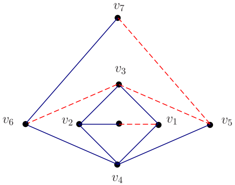
Contract $B \cup \{d\}$ to a vertex d_1; contract $\bar{B} \cup \{d\}$ to a vertex d_2.

Problems and following work

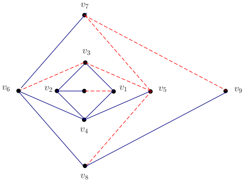
The sequence of signed bipartite planar graphs with $\chi_c = 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$:



(e) Ω_3



(f) Ω_5



(g) Ω_7

The only know examples are graphs of girth exactly 4.

Problems and following work

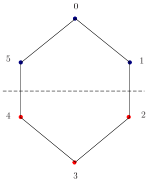
Conjecture 38

There exists α such that for every 5-edge-connected signed planar graph (G, σ), $\Phi_c(G) \leq \alpha < 4$.

Conjecture 39

For every 5-edge-connected signed planar graph (G, σ), $\Phi_c(G) \leq 3$.

This is equivalent to a problem of $\mathbb{Z}_6$-flow, which is stronger than $\mathbb{Z}_3$-connected. Moreover, we reckon the two conjectures are equivalent in some way.



And other signed graphs besides planar graphs?

Theorem 40

Every 5-edge-connected projective planar graph is $\mathbb{Z}_3$-connected.

J. V. de Jong, R. B. Richter, Strong 3-Flow Conjecture for Projective Planar Graphs, Journal of Graph Theory, 2022+.

Theorem 41

Every 4-edge-connected toroidal graph admits a nowhere-zero 3-flow.

J. Li, Y. Ma, Z, Miao, Y. Shi, W. Wang, C. Q. Zhang, Nowhere-zero 3-flows in toroidal graphs, Journal of Combinatorial Theory, Series B, 153 (2022) 61-80.

Thanks!