Non-repeated cycle lengths and Sidon sequences

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Overview

Introduction

See Bondy and Murty, Graph Theory with Applications, p.247, Problem 11.

Open Problem 1(Erdős, 1975)

Let $n + f(n)$ be the maximum possible number of edges in a graph on n vertices in which no two cycles have the same length. Determine $f(n)$.

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Let $n+f(n)$ be the maximum possible number of edges in a graph on n vertices in which no two cycles have the same length. Determine $f(n)$.

- $f(n) \geq \lfloor(\sqrt{8n-15}-3)/2\rfloor$. (Shi 1988)
- $f(n) \geq \sqrt{238n/99} \approx 1.55\sqrt{n}$. (Lai 2020)
- $f(n) \leq 1.98\sqrt{n}$. (Boros, Caro, Füredi and Yuster 2001)

$$1.55\sqrt{n} \leq f(n) \leq 1.98\sqrt{n}.$$

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Proof:

- Set $e = n + f(n)$.
- Let B be a set of edges chosen independently and uniformly at random from $E(G)$ with probability $1/\sqrt{e}$.
- Let $g(B)$ be the number of cycles in $G - B$.
- By deleting one edge in each cycle, we can get a graph with at most $n - 1$ edges.
- Thus $e - |B| - g(B) \leq n - 1$ for any B.

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- So $\mathbb{E}[g(B)] = \sum_{k=3}^{+\infty}(1 - 1/\sqrt{e})^k < \sqrt{e}$.

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- A cycle $C_k \subseteq G - B$ with probability at most $(1 - 1/\sqrt{e})^k$.
- So $\mathbb{E}[g(B)] = \sum_{k=3}^{+\infty}(1 - 1/\sqrt{e})^k < \sqrt{e}$.
- $\mathbb{E}[|B| + g(B)] = \mathbb{E}[|B|] + \mathbb{E}[g(B)] < 2\sqrt{e}$.
- We get $e \leq n - 1 + 2\sqrt{e} \leq n + 2\sqrt{n}$.

$\square$

Introduction

Another interesting problem is to consider the restricted version of Erdős' problem for 2-connected graphs.

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let $n + f_2(n)$ be the maximum number of edges in an n-vertex 2-connected graph in which no two cycles have the same length. Determine $f_2(n)$.

- $f_2(n) \leq \sqrt{2n} + o(\sqrt{n})$. (Shi, 1988)
- $f_2(n) \geq \sqrt{n/2} - o(\sqrt{n})$. (Chen, Lehel, Jacobson and Shreve, 1998)
- $f_2(n) \geq \sqrt{n} - o(\sqrt{n})$. (Boros, Caro, Füredi and Yuster, 2001)

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- G has at least $\frac{1}{2}(f_2(n)+1)(f_2(n)+2)$ cycles.

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- G has at least $\frac{1}{2}(f_2(n)+1)(f_2(n)+2)$ cycles.
- Since G has at most n cycles, then $f_2(n) \leq \sqrt{2n} + o(\sqrt{n})$.

□

The lower bound

Definition

A sequence of integers $a_1, a_2, \ldots, a_k$ is called a **Sidon sequence** (or Sidon set, B_2-set) if all pairwise sums $a_i + a_j$ for $1 \leq i \leq j \leq k$ are distinct.

- Let $S(n)$ denote the maximum size of a Sidon subsequence of $\{1, 2, \ldots, n\}$.
- It is well known that $S(n) = \sqrt{n} + o(\sqrt{n})$.
- The upper bound was proved by Erdős and Turán.
- The lower bound was provided by Singer.

The lower bound

Theorem 3 (Boros, Caro, Füredi and Yuster)

$$f_2(n) \geq \sqrt{n} - O(n^{9/20}).$$

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Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(G)$ consist of the edges in a Hamilton cycle $C = v_0 v_1 ... v_{n-1} v_0$ and the edges $v_0 v_{a_i}$ for all $1 < i < k$.

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- Each cycle in G contains two edges incident to v_0 (say $v_0 v_{a_i}$ and $v_0 v_{a_j}$) and the subpath of C between v_{a_i} and v_{a_j} not containing v_0.
- All cycle lengths in G are of the form $a_j - a_i + 2$ for $1 \leq i < j \leq k$, which are pairwise distinct.

□

Our main result

Conjecture 4 (Boros, Caro, Füredi and Yuster, 2001)

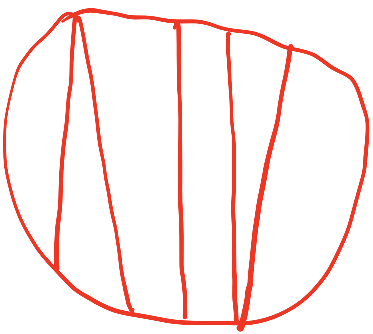
$$\lim_{n\to\infty} f_2(n)/\sqrt{n} = 1.$$

Theorem 5 (Ma and Yang, 2021)

$$f_2(n) = \sqrt{n} + o(\sqrt{n}).$$

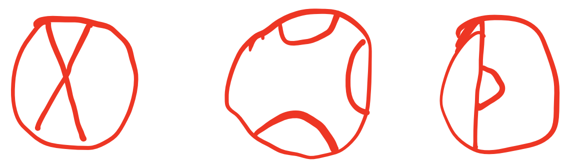
A simple case

In the following case, all paths are '**well ordered**'. Then we will prove that $s = f_2(n) \leq \sqrt{n} + o(\sqrt{n})$.



all paths are parallel

no :



The paths are $g_0, g_2, \cdots, g_s$. By $g_i \Delta g_j$ we denote the subgraph consisting of the edges which appears in exactly one of g_i and g_j. We say $g_i \Delta g_j$ is a cycle of order $|j-i|$.

A simple case

Let L be the sum of length of cycles with order at most k.
On the one hand:

- There are $(s-r)$ cycles with order r.
- The number of cycles with order at most $k(=o(s))$ is $M=\sum_{r=1}^{k}(s-r)\approx sk$.
- These cycles have different length. So $L\geq 1+\cdots+M\approx\frac{1}{2}s^2k^2$.

A simple case

On the other hand:

- Each edge is contained in at most r cycles of order r, for $r \geq 2$.
- The sum of length of cycles with order r is at most $r(1 + o(1))n$.
- Totally, we have $L \leq \sum_{r=1}^{k} r(1 + o(1))n \approx \frac{1}{2}k^2 n$.

Thus we have

$$\frac{1}{2}s^2 k^2 \leq L \leq \frac{1}{2}k^2(1 + o(1))n,$$

and

$$s \leq (1 + o(1))\sqrt{n}.$$

□

Difficulities

Proof ideas 1

Ear-decomposition, linear order and paths:
Firstly, we get ears $P_1, \cdots, P_s$. Secondly, we define a linear order on V by tree L. Finally, we get s different $u - v$ paths f_i with $P_i \subseteq f_i$.

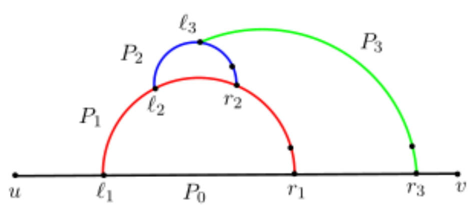


Figure 1 (a). Ear decomposition

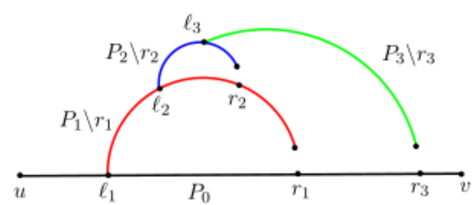


Figure 1 (b). Tree L

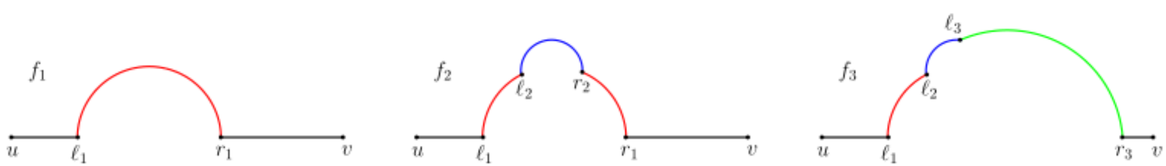


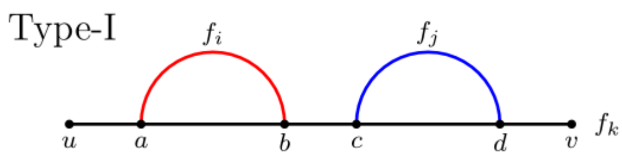
Figure 1 (c). Paths f_1, f_2 and f_3

Proof ideas 1

Lemma 6

For distinct $i, j \in \{0, 1, ..., s\}$, $f_i \Delta f_j$ consists of one or two cycles.

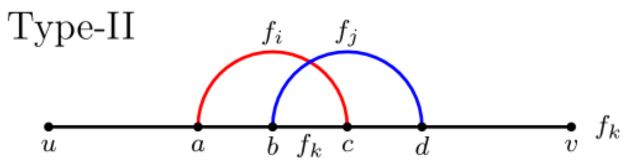
- A pair $\{f_i, f_j\}$ is called **type-I**, if $f_i \triangle f_j$ consists of two cycles.



In this case, we find a path f_k as the **base** of $\{f_i, f_j\}$.

Proof ideas 1

- A pair $\{f_i, f_j\}$ is called **type-II**, if it is not type-I and there exists some f_k such that $a < b < c < d$ lie in f_k.



Such a path f_k is called a **crossing path** of $\{f_i, f_j\}$, and the crossing path f_k with minimum k is called the **base** of $\{f_i, f_j\}$

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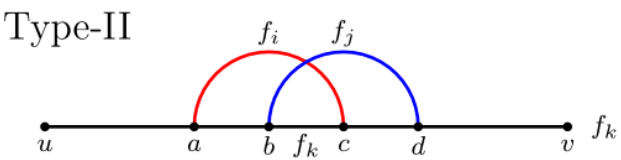

Such a path f_k is called a **crossing path** of $\{f_i, f_j\}$, and the crossing path f_k with minimum k is called the **base** of $\{f_i, f_j\}$.

- Finally, a pair $\{f_i, f_j\}$ is **normal**, if it is neither type-I nor type-II.

Proof ideas 2

Almost all pairs are normal.

Lemma 8

There exist disjoint set of paths $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ such that $\sum_{i\in[4]} |\mathcal{F}_i| \geq s - 90\sqrt{n}/\log n$, and each $\mathcal{F}_i$ contains at most $2\sqrt{n}\log^2 n$ pairs of type-I and type-II.

Otherwise we will get more than n cycles.

Proof ideas 3

Reordering and partitioning $\mathcal{F}$:
We can reorder most paths and partition them into a bounded number of intervals such that for every relevant edge e, paths containing e in each interval are listed almost consecutively.

More results on Sidon sequence

Sidon's problem has many remarkable connections to Fourier anyalysis, abstract algebra, coding theory and extremal graph theory. It is a wonderful unity of mathematics.

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A sequence of integers $a_1, a_2, ..., a_k$ is called a **Sidon sequence** (or Sidon set, B_2-set) if all pairwise sums $a_i + a_j$ for $1 \leq i \leq j \leq k$ are distinct.

- Let $S(n)$ denote the maximum size of a Sidon subsequence of $\{1, 2, ..., n\}$.
- $S(n) > \sqrt{n}$ infinitely many times. (Singer, 1938)
- $S(n) \leq \sqrt{n} + O(n^{1/4})$. (Erdős and Turán, 1941)
- $S(n) \leq \sqrt{n} + n^{1/4} + 1$. (Lindström, 1969)
- $S(n) \leq \sqrt{n} + n^{1/4} + 1/2$. (Chilleruelo, 2010)

More results on Sidon sequence

Open Problem 3 (Erdős, \$ 500)

Prove or disprove that for every $\varepsilon > 0$ the equality $S(n) < \sqrt{n} + o(n^{\varepsilon})$ holds.

Putting the two methods together, they get

Theorem (Balogh, Furedi and Roy, 2021)

There exists a constant $\gamma \geq 0.002$ and a number n_0 such that for every $n > n_0$

$$S(n) \leq \sqrt{n} + n^{1/4}(1 - \gamma).$$

Thank you!

Thank you very much for your attention!