Counterexamples for chromatic and dichromatic number

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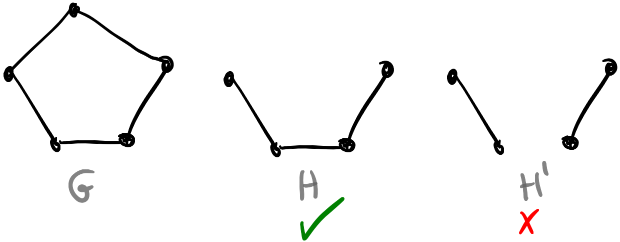
Induced subgraphs

Let G and H be graphs.

H is an **induced subgraph** of G if H can be obtained from G by deleting vertices.

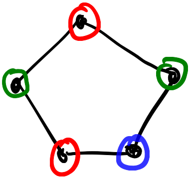
If H is not an induced subgraph of G, then G is **H-free**.



Colouring

The **chromatic number** $\chi(G)$ is the smallest k such that G has a k-**colouring**: a function $f : V(G) \to \{1, \dots, k\}$ with $f(u) \neq f(v)$ for all $uv \in E(G)$.

The **clique number** $\omega(G)$ is the size of a largest clique (a set of vertices all pairwise adjacent) in G.



G

$\chi(G) = 3$
$\omega(G) = 2$

Questions

- What can we say about graphs G with $\chi(G)$ much larger than $\omega(G)$?

A class $\mathcal{C}$ of graphs is polynomially χ-**bounded** if there is a polynomial function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

- Which hereditary classes are χ-bounded? [Gyárfás]

closed under
induced subgraphs

The conjecture

- Maybe a class is χ-bounded if its triangle-free graphs have bounded χ?

Conjecture (Origin unclear; sometimes attributed to Esperet)

For all $k, r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that for every graph G with $\chi(G) \geq n$ and $\omega(G) \leq k$, there is an induced subgraph H of G with $\chi(H) \geq r$ and $\omega(H) = 2$.

- True if we omit "induced" [Rödl]

The counterexample

Theorem (Carbonero, Hompe, Moore, S.)

For every $n \in \mathbb{N}$, there is a graph G with $\chi(G) \geq n$ and $\omega(G) \leq 3$ such that every induced subgraph H of G with $\omega(H) \leq 2$ satisfies $\chi(H) \leq 4$.

3?

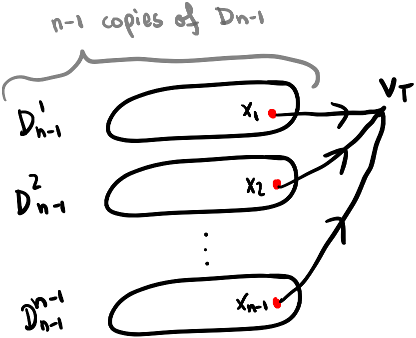
Theorem (Scott, Seymour)

For every $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that every graph G with $\chi(G) \geq n$ and $\omega(G) \leq k$ contains an induced subgraph H with $\omega(H) \leq 2$ and $\chi(H) = 3$.

odd hole

Zykov graphs

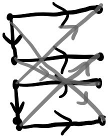
Define D_n as follows: $D_1 = \bullet$



For every $T=(x_1,\dots,x_{n-1})$ with $x_i \in D^i_{n-1}$, add a vertex v_T and edges $x_i \to v_T$ for all i.

D_2:

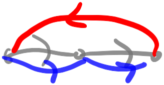
D_3:



- $\chi(D_n) \geq n$. [Zykov]
- For all $u, v \in D_n$, there is at most one directed path between them. [Kierstead, Trotter] ~ implies: acyclic

Modification

Given D_n, define D'_n:

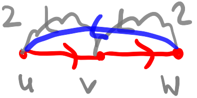


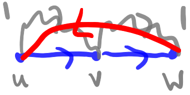
- $V(D'_n) = V(D_n)$;
- if there is a path from u to v of length congruent to 1 modulo 3, add a blue edge $u \to v$;
- if there is a path from u to v of length congruent to 2 modulo 3, add a red edge $v \to u$.

Then:

- D_n is a subgraph of D'_n (so $\chi(D'_n) \geq n$).
- If $u \to v$ then $v \not\to u$.
- If $u \to v$ and $v \to w$ in red, then $w \to u$ in blue.

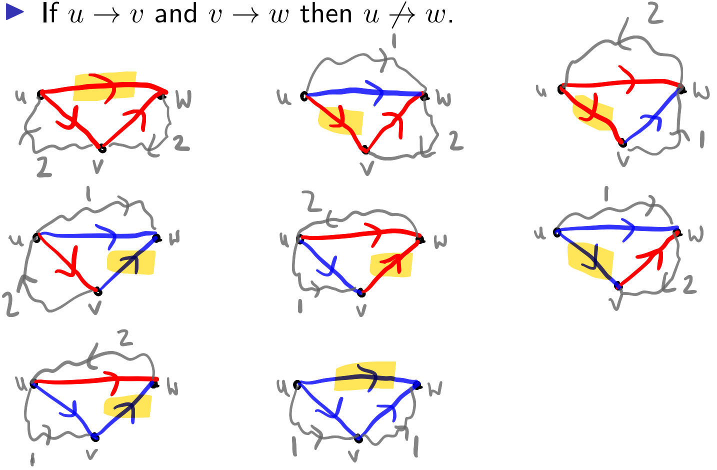
$u \to v$ in D_n is a path of length 1.





Clique number

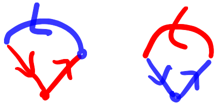
- If $u \to v$ and $v \to w$ then $u \not\to w$.



If 4 vertices are pairwise adjacent, one has ≥ 2 out-nbrs.

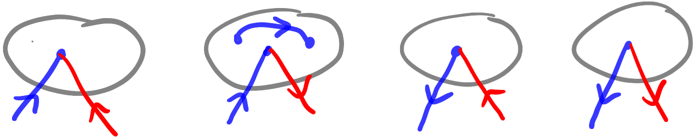
Chromatic number of triangle-free induced subgraphs

- If H is an induced subgraph of D'_n with $\omega(H) \leq 2$, then $\chi(H) \leq 4$.



no vertex has an in-edge an out-edge of the same colour

Group and notice that each is stable:



Generalizations

Theorem (Briański, Davies, Walczak)

For every prime $p \geq 2$ and every $n \in \mathbb{N}$, there is graph D_n^p with $\chi(D_n^p) \geq n$ and $\omega(D_n^p) = p$; and $\chi(H) \leq \binom{\omega(H)+2}{3}$ for every induced subgraph H of D_n^p with $\omega(H) < p$.

disproves a conjecture of Esperet that
χ-bounded $\Rightarrow$ polynomially χ-bounded

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Theorem (Girão, Illingworth, Powierski, Savery, Scott, Tamitegama, Tan)

For every graph F with at least one edge, there is a constant c_F such that there are graphs of arbitrarily large chromatic number and the same clique number as F in which every F-free induced subgraph has chromatic number at most c_F.

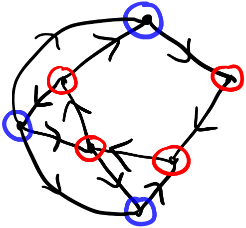
(in our result, $F = \Delta$.

Dichromatic number

Let D be a digraph.

A **k-dicolouring** of D is a function $f : V(D) \to \{1, \ldots, k\}$ such that no directed cycle is monochromatic. [Erdős, Neumann-Lara]

The **dichromatic number** $\vec{\chi}(D)$ is the minimum k such that D has a k-dicolouring.



How can we construct a digraph with large $\vec{\chi}$ and small ω?

→ randomly [Harutyunyan, Mohar]

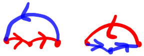
→ explicitly [Alon]

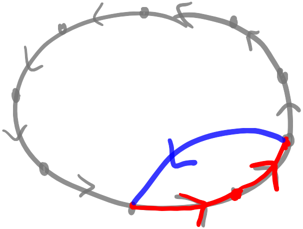
Dichromatic number

Theorem (Carbonero, Hompe, Moore, S.)

$\vec{\chi}(D_n') \geq n/4$ and D_n' has no odd induced directed cycle of length at least 5.

- If $H \subseteq D_n'$ is acyclic, then H is triangle-free so $\chi(H) \leq 4$. Thus $\chi(D_n') \leq 4\vec{\chi}(D_n')$.





An odd cycle contains two consecutive edges of the same colour.

Dichromatic number

Theorem (Carbonero, Hompe, Moore, S.)

$\vec{\chi}(D'_n) \geq n/4$ and D'_n has no odd induced directed cycle of length at least 5.

- Call a digraph t-**chordal** if every induced directed cycle has length t.

Theorem (Aboulker, Bousquet, de Verclos)

For every n, there is a 3-chordal digraph G_n with $\vec{\chi}(G_n) \geq n$ and $\omega(G_n) = 3$.

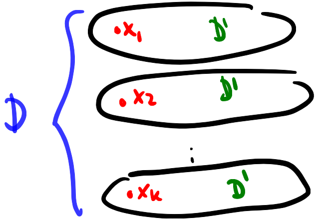
Theorem (Carbonero, Hompe, Moore, S.)

For every $t \geq 4$ and every n, there is a t-chordal digraph G_n with $\vec{\chi}(G_n) \geq n$ and $\omega(G_n) = 2$.

t-chordal digraph construction

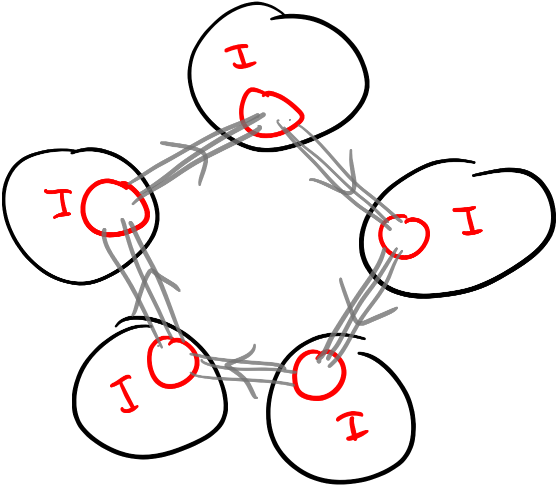
Main idea: [Aboulker, Bousquet, de Verclos]

- Let D' be a t-chordal digraph with $\vec{\chi}(D') = k$. Let D consist of k disjoint copies of D'
- Let $\mathcal{I}$ be the set of all k-vertex independent sets of D with one vertex in each copy of D'
- Construct a digraph G consisting of disjoint copies of D such that in every k-dicolouring, in one copy of D, no $I \in \mathcal{I}$ sees all k colours
- It follows that G has $\vec{\chi}(G) \geq k + 1$



t-chordal digraph construction

One stable set $I \in \mathcal{I}$:

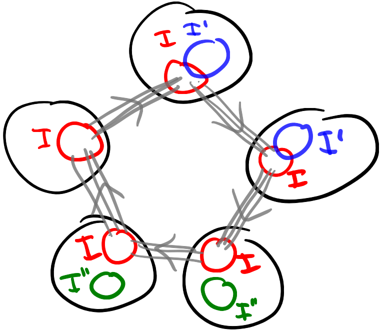


t copies & create a blow-up of a t-cycle from copies of I

t-chordal digraph construction

One stable set $I \in \mathcal{I}$:

If each $I^* \in \mathcal{I}^*$ misses a colour, then in one of the five sets, every $I^{**} \in \mathcal{I}$ misses a colour.



Now apply induction to this graph D^* and $\mathcal{I}^*$, defined as follows:

- if $I' \in \mathcal{I}$ is disjoint from I, then add the union of all t copies of I' as a new set to $\mathcal{I}^*$;
- otherwise, for $I' \neq I$, add t new sets to $\mathcal{I}^*$, one for each copy of I'.

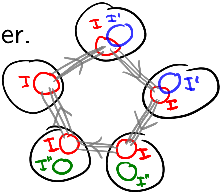
t-chordal digraph construction

Now apply induction to this graph D^* and $\mathcal{I}^*$, defined as follows:

- if $I' \in \mathcal{I}$ is disjoint from I, then add the union of all t copies of I' as a new set to $\mathcal{I}^*$;
- otherwise, for $I' \neq I$, add t new sets to $\mathcal{I}^*$, one for each copy of I'.

What got better?

- The intersection graph of $\mathcal{I}^*$ has at most the same chromatic number as that for $\mathcal{I}$;
- The colour class that used to contain I got smaller.

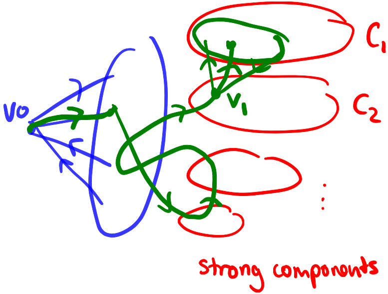


No directed P_k, $C_{<k}$

Theorem (Carbonero, Hompe, Moore, S.)

For every $k \geq 3$, the class of digraphs with no monotone induced path on k vertices, and no induced directed cycle on fewer than k vertices is $\vec{\chi}$-bounded.

Gyárfás path argument, replacing "connected" by "strongly conn.":



Repeating this gives a green path which is forward-induced and v_0 has no nbrs after v_1

No directed P_k, $C_{<k}$

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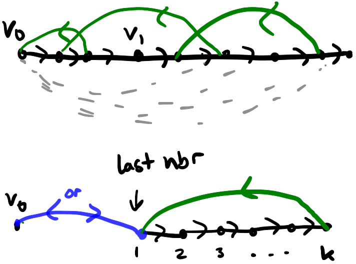

- Forward-induced
- For every subpath on k vertices, there is a back-edge making it into a cycle

The digraph Gyárfás-Sumner conjecture

need to exclude a forest
[Harutyunyan, Mohar]

Conjecture (Aboulker, Charbit, Naserasr)

For every directed tree T, the class of T-free digraphs is $\vec{\chi}$-bounded.

- True for stars and $\to\to\leftarrow$ [Chudnovsky, Scott, Seymour]
- True for $\to\leftarrow\to$ when $\omega \leq 3$ [Steiner]
- Open for monotone P_4! True for $\omega \leq 4$ [Aboulker, Charbit, Naserasr]

not χ-bounded [Kierstead, Trotter]

Thank you!

Questions?