On the sum of the first two largest signless Laplacian eigenvalues

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Outline

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Introduction

- Let $A(G)$ and $D(G)$ be the adjacency matrix and diagonal matrix vertex degrees of graph G, respectively.
- The Laplacian matrix and signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively.
- The eigenvalues of $L(G)$ and $Q(G)$ are called Laplacian eigenvalues and signless Laplacian eigenvalues of G, respectively, and are denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$, respectively.

Introduction

- A natural and fundamental problem in spectral graph theory is the relationship between the eigenvalues of a graph and its structural parameters.
- $\sum_{i=1}^{n} \mu_i(G) = \sum_{i=1}^{n} q_i(G) = 2e$, where n and e are the order and size of G, respectively.
- $\sum_{i=1}^{k} \mu_i(G)$ or $\sum_{i=1}^{k} q_i(G)$ for $1 \le k \le n-1$? .

Introduction

- Some results and conjectures related to $\sum_{i=1}^{k} \mu_i(G)$ can be found in the literature. First we state the Grone-Merris conjecture. For a graph G with degree sequence $\{d(v)|v \in V(G)\}$, the following holds.

Conjecture 1.1 (Grone-Merris)

For any graph G with n vertices and for any $k \in \{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{k} \mu_i(G) \leq \sum_{i=1}^{k} |\{v \in V(G)|d(v) \geq i\}|.$$

- This conjecture was proved by Hua Bai and now is called the Grone-Merris theorem.

R. Grone, R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.*, **7**(1994), 221–229.

H. Bai, The Grone–Merris conjecture, *Trans. Amer. Math. Soc.*, **363**(2011), 4463–4474.

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- As a variation on the Grone–Merris conjecture, Brouwer proposed the following conjecture for Laplacian eigenvalues.

Conjecture 1.2 (Brouwer)

For any graph G with n vertices and for any $k \in \{1, 2, \dots, n\}$,

$$\sum_{i=1}^{k} \mu_i(G) \leq e(G) + \binom{k+1}{2}.$$

- By using computer computations, Brouwer has checked Conjecture 1.2 for all graphs with at most 10 vertices.

A.E. Brouwer, W.H. Haemers, Spectra of graphs, Springer, New York, 2012.

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- For $k = 1$, the Conjecture 1.2 follows from the well-known inequality $\mu_1(G) \leq n \leq e(G) + 1$.

- For $k = n$ and $k = n - 1$, the Conjecture 1.2 follows trivially from the fact that $\sum_{i=1}^{n-1} \mu_i(G) = \sum_{i=1}^{n} \mu_i(G) = 2e(G) \leq e(G) + \binom{n}{2}$.

- Haemers et al. showed that Conjecture 1.2 is true for $k = 2$, that is $\mu_1(G) + \mu_2(G) \leq e(G) + 3$ for any graph G.

W.H. Haemers, A. Mohammadian, B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **432**(2010), 2214–2221.

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- Moreover, the Conjecture 1.2 was proved to be true for several classes of graphs (for all $k \in \{1, 2, \ldots, n\}$) such as trees, threshold graphs, unicyclic graphs, bicyclic graphs, regular graphs and split graphs.

X. Chen, Improved results on Brouwer's conjecture for sum of the Laplacian eigenvalues of a graph, *Linear Algebra Appl.*, **557**(2018), 327–338.

X. Chen, J. Li, Y. Fan, Note on an upper bound for sum of the Laplacian eigenvalues of a graph, *Linear Algebra Appl.*, **541**(2018), 258–265.

W. Li, J. Guo, On the full Brouwer's Laplacian spectrum conjecture, *Discrete Mathematics.*, **345**(2022)113078.

Introduction

- Let $S_k(G) = \sum_{i=1}^{k} q_i(G)$ be the sum of the first k largest signless Laplacian eigenvalues of G.

Conjecture 1.3 (Ashraf)

For any graph G with n vertices and for any $k \in \{1, 2, \dots, n\}$,

$$S_k(G) \leq e(G) + \binom{k+1}{2}.$$

- By using computer computations, Ashraf et al. has checked Conjecture 1.3 for all graphs with at most 10 vertices.

F. Ashraf, G.R. Omidi, B. Tayfeh-Rezaibe, On the sum of signless Laplacian eigenvalues of graph, *Linear Algebra Appl.*, **438**(2013), 4539-4546.

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- Ashraf et al. proved that Conjecture 1.3 is true for $k = 2$, that is $S_2(G) \leq e(G) + 3$ for any graph G. But the key lemma they used is incorrect which has a counterexample.
- Zheng proved that Conjecture 1.3 is true for all connected triangle-free graphs when $k = 2$.

F. Ashraf, G.R. Omidi, B. Tayfeh-Rezaibe, On the sum of signless Laplacian eigenvalues of graph, *Linear Algebra Appl.*, **438**(2013), 4539-4546.

Y. Zheng, A note on the sum of the two largest signless Laplacian eigenvalues, *Ars Combin.*, **148**(2020), 183–191.

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- Therefore, Conjecture 1.3 is still open when $k = 2$.
- We prove that $S_2(G) < e(G) + 3$ is true for any graphs which also confirm the conjecture 1.3 when $k = 2$.

Theorem 1.1 (Zhou, He)

For any graph G with n vertices,

$$S_2(G) < e(G) + 3.$$

Introduction

- Ashraf et al. proved that Conjecture 1.3 is true for all graphs when $k \in \{1, n-1, n\}$, and for regular graphs(for all k).
- Yang and You proved that Conjecture 1.3 is true for unicyclic graphs and bicyclic graphs (for all k).
- For more details, we refer to:

F. Ashraf, G.R. Omidi, B. Tayfeh-Rezaibe, On the sum of signless Laplacian eigenvalues of graph, *Linear Algebra Appl.*, **438**(2013), 4539-4546.

J. Yang, L. You, On a conjecture for signless Laplacian eigenvalues, *Linear Algebra Appl.*, **446**(2014), 115–132.

X. Chen, G. Hao, D. Jin, J. Li, Note on a conjecture for the sum of signless Laplacian eigenvalues, *Czech Math J.*, **68**(2018), 601–610.

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The proof of our main result

- This lemma is the key to our approach. It gives a sufficient condition for the truth of Theorem3.1, that holds for almost all graphs.

Lemma 2.1 (Zhou, He)

If G is a graph with a nonempty subgraph H for which $S_2(H) \le e(H)$, then $S_2(G) < e(G) + 3$.

- Noting that for $H = 4K_2$ or $H = 3K_{1,2}$, one has $S_2(H) = e(H)$, we may assume G contains neither $H = 4K_2$ nor $H = 3K_{1,2}$ as a subgraph.
- It is sufficient to consider only graphs G whose matching number $m(G)$ is at most 3.
- We prove Theorem 3.1 for $m(G) = 1, 2$ and 3, respectively.

The proof of our main result

Proof.

Case 1. $m(G) = 1$. It is easy to check that either $G = K_{1,k-1} \cup (n-k)K_1$ for some $1 \leq k \leq n$ or $G = K_3 \cup (n-3)K_1$, the assertion holds.

Case 2. $m(G) = 2$. We may assume that G is a connected graph with at least 11 vertices. First suppose that G has no K_3 as a subgraph, since $m(G) = 2$, G has no odd cycle as a subgraph. It is obvious that G is a bipartite graph, the assertion holds. ∎

The proof of our main result

Proof.

Next suppose that G has a subgraph $H = K_3$.

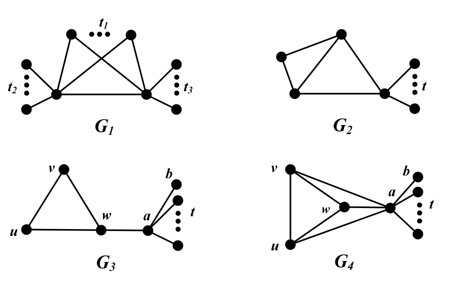

Figure: Possible forms of graphs G with $m(G) = 2$

∎

The proof of our main result

Lemma 2.2 (Fiedler)

If $\lambda_1,\ \lambda_2,\ldots,\lambda_n$ are all eigenvalues of A, then the eigenvalues of $\Delta_k(A)$ are the $\binom{n}{k}$ distinct sums of the λ_i taken k at a time.

Corollary 2.1 (Zhou, He)

If $q_1,\ q_2,\ldots,q_n$ are signless eigenvalues of G, and $P(Q^\pi,x)$ contains the largest two roots of $P_Q(G,x)$, then the largest eigenvalue of $\Delta_2(Q^\pi)$ equals $S_2(G)$.

Proof.

Case 3. $m(G)=3$. ∎

M. Fiedler, Special Matrices and their Applications in Numerical Mathematics, Martinus Nijhoff, Dordrecht, 1984.

Outline

A remark

Theorem 3.1 (Zhou, He)

For any graph G with n vertices,

$$S_2(G) < e(G) + 3.$$

Remark 1

Ashraf et al. proved that

$$S_k(G) \leq e(G) + \binom{k+1}{2}$$

is asymptotically tight for any k for the graph $K_k \vee \overline{K_t}$, the join of K_k and the empty graph $\overline{K_t}$.

F. Ashraf, G.R. Omidi, B. Tayfeh-Rezaibe, On the sum of signless Laplacian eigenvalues of graph, *Linear Algebra Appl.*, **438**(2013), 4539-4546.

Thank you!