# Upper bounds on the bichromatic number of some graphs

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- Basic definitions

- Basic definitions

- Known results

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- Our results

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- The proof

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- Open problems

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- The **cochromatic number** of $G$: the minimum number that is required to partition the vertex set of a graph into that many sets, each of which being either an **independent set** or a **clique**.

  [Lesniak and Straight (1977)]

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$$S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$$

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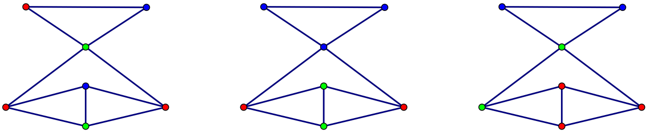
- **Example**.



$(3, 0)$-colorable $\quad$ $(1, 2)$-colorable $\quad$ $(0, 3)$-colorable

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- $(0,l)$-colorable graphs are exactly those graphs of clique covering number at most $l$.

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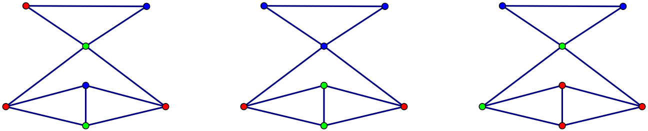
- **An example for the case** $\chi^b(G) = 4$. Not $(2,1)$-colorable!



$(3,0)$-colorable $(1,2)$-colorable $(0,3)$-colorable

$\Rightarrow$ $(4,0)$-colorable,$(3,1)$-colorable,$(2,2)$-colorable,$(1,3)$-colorable,$(0,4)$-colorable.

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- Upper bound:

$$\chi^b(G) \leq \chi(G) + \theta(G) - 1.$$

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**Proof:** If $k + l = \chi(G) + \theta(G) - 1$, then $k \geq \chi(G)$ or $l \geq \theta(G)$.

It follows that $G$ is $(k, l)$-colorable. $\square$

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- **Complete $n$-partite graph**: a $n$-partite graph (i.e., a set of graph vertices admits a partition into $n$ classes s.t. no two vertices within the same class are adjacent) s.t. every pair of vertices from different classes are adjacent.

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- Let $K_{p_1,p_2,\ldots,p_n}$ be the **complete $n$-partite graph** with $p_i$ vertices in the $i$-th partite set, $1 \leq i \leq n$, $p_0 = 0 < p_1 \leq p_2 \leq \ldots \leq p_n$.

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**Proposition (Lesniak and Straight, Ars Combin., 1977)**

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- $\chi^b(\overline{G}) \le n - k + p_k$.
  (Since $\overline{G}$ is $(p_i, n - i)$-colorable, $\overline{G}$ is $(n - k + p_k - (n - i), n - i)$-colorable.)

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  (Since $\overline{G}$ is $(p_i, n-i)$-colorable, $\overline{G}$ is $(n-k+p_k-(n-i), n-i)$-colorable.)
- $\chi^b(\overline{G}) > n - k + p_k - 1$.
  (If $1 \le k \le n$, then $\overline{G}$ is not $(p_k - 1, n-k)$-colorable.
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- $\chi^b(G) = \chi^b(\overline{G}) = \max\ \{n - i + p_i \mid 0 \le i \le n\}$.

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**Proposition (Epple and Huang, JGT, 2010)**

*For every graph $G$ on $n$ vertices, $\chi^b(G) \geq \sqrt{n}$.*

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- **Proof:** $\chi^b(G) \cdot \chi^b(G) \geq \chi(G) \cdot \theta(G) \geq \chi(G) \cdot \alpha(G) \geq n$. $\square$

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**Theorem (Epple and Huang, JGT, 2010)**

*The problem of computing the bichromatic number of a graph is NP-hard.*

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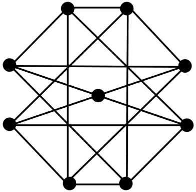
**Theorem (Epple and Huang, JGT, 2014)**

*For any graph $G$,*

$$\chi^b(G) \leq \Delta^b(G) + 1.$$

*Equality holds iff $G$ is one of $K_n, K_{m,m}, C_5, Q$ or their complements.*

- The graph $Q$ in the above theorem is depicted below.



The class of **cograph** is recursively defined as follows:

(i) $K_1$ is a cograph;

(ii) if $G$ is a cograph, then $\overline{G}$ is a cograph;

(iii) if $G$, $H$ are cographs, then disjoint union of $G$ and $H$ is a cograph.

[Corneil, Lerchs and Burlingham (1981)]

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**box cograph**: a box cograph is a cograph $G$ having exactly $\chi(G)\theta(G)$ vertices.

The class of box cographs $G$ is denoted by $\mathcal{B}(r,s)$ if $\chi(G) = r$ and $\theta(G) = s$.

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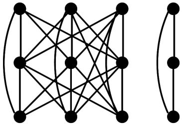
- **Example:** a box cograph of dimension 3 by 4.



**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

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- $G$ is not $(k-1, l-1)$-colorable.
  ($\chi^b(G) > k + l - 2$. $G$ is not $(k', l')$-colorable for some $k' + l' = k + l - 2$. Since $G$ is $(k, 0)$-colorable and $(0, l)$-colorable, $k' \leq k - 1$ and $l' \leq l - 1$. Thus $k' = k - 1$ and $l' = l - 1$.)

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- $\mid S_i \cap C_j \mid = 1$. $(\mid S_i \cap C_j \mid \leq 1$. Suppose $\mid S_i \cap C_j \mid = 0$ for some $i$ and $j$. Then $\{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k, C_1 \cap S_i, \ldots, C_{j-1} \cap S_i, C_{j+1} \cap S_i, \ldots, C_l \cap S_i\}$ is a $(k-1, l-1)$-coloring of $G$.)

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- $G$ has $kl$ vertices.
- $G$ is a cograph.

  (Frequently employ the property: if $G$ is a cograph with at least two vertices then either $G$ or $\overline{G}$ is disconnected.) $\square$

## 2. Known results

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$ and $\theta(G) = l$. Then $\chi^b(G) \leq k + l - 1$, and the following statements $(i), (ii)$ and $(iii)$ are equivalent:*

(i) $\chi^b(G) = k + l - 1$,

(ii) *$G$ is not $(k-1, l-1)$-colorable,*

(iii) $G \in \mathcal{B}(k, l)$.

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- (*ii*) *$G$ is not $(k-1, l-1)$-colorable,*
- (*iii*) $G \in \mathcal{B}(k, l)$.

- **An example for the case $\chi^b(G) - \theta(G)$ arbitrarily large.**

  Given positive integers $m$ and $n$, let $mK_n$ denote the disjoint union of $m$ copies of $K_n$. It is clear that $mK_n \in \mathcal{B}(n, m)$, and thus

$$\chi^b(mK_n) = \chi(G) + \theta(G) - 1 = n + m - 1.$$

# 3. Our results

The **clique number** $\omega(G)$: the maximum order over all cliques of $G$.

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A natural upper bound: $\chi^b(G) \leq \chi(G) + \theta(G) - 1$.

### Theorem

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and the following statements $(i), (ii), (iii)$ and $(iv)$ are equivalent:*

- $(i)$ $\chi^b(G) = \theta(G) + 1$,
- $(ii)$ *$G$ is not $(1, \theta(G) - 1)$-colorable,*
- $(iii)$ $G \in \mathcal{B}(2, \frac{|V(G)|}{2})$,
- $(iv)$ *$G$ is the disjoint union of balanced complete bipartite graphs.*

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## Theorem

*Let $G$ be a graph with $\omega(G) < 4$. Then $\chi^b(G) \leq \theta(G) + 2$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + 2$,

(ii) *$G$ is not $(2, \theta(G) - 1)$-colorable,*

(iii) $G \in \mathcal{B}(3, \frac{|V(G)|}{3})$.

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**Theorem**

*Let $G$ be a graph with $\omega(G) < 4$. Then $\chi^b(G) \leq \theta(G) + 2$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + 2$,

(ii) *$G$ is not $(2, \theta(G) - 1)$-colorable,*

(iii) $G \in \mathcal{B}(3, \frac{|V(G)|}{3})$.

- Remark: If $\omega(G) < 4$, then $\chi^b(G) \leq \theta(G) + \omega(G) - 1$.

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- Problem: If $\omega(G) \geq 4$ ?

## Theorem

*Let $G$ be a line graph of a simple graph with $\omega(G) < r + 1$, $r \geq 4$. Then $\chi^b(G) \leq \theta(G) + r - 1$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + r - 1$,

(ii) *$G$ is not $(r - 1, \theta(G) - 1)$-colorable,*

(iii) *$G$ is the disjoint union of $K_r$.*

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

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*If $G$ is triangle free graph other than $K_2$, then $\chi(G) = \chi^c(G)$.*

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Since $(C_{l-1} \cup C_l) \subseteq V(G_i)$, $G_i \neq K_2$. By Lemma,

$$\chi(G_i) = \chi^c(G_i) \leq \theta(G_i) = l - i.$$

Then $V(G_i)$ can be partitioned into $l - i$ independent sets

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i.e., $G$ is $(l - i, i)$-colorable for each $i \in \{0, 1, \ldots, l - 2\}$., and

$G$ is $(l - i + 1, i)$-colorable for each $i \in \{0, 1, \ldots, l - 1\}$.

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If $G$ is $(1, l-1)$-colorable, $\chi^b(G) \leq l$ since $G$ is $(0, l)$-colorable.

If $G$ is not $(1, l-1)$-colorable, $\chi^b(G) = l + 1$ since $G$ is $(1, l)$-colorable and $(0, l+1)$-colorable.

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This completes the proof. □

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**Theorem**

*Let $G$ be a triangle free graph on $n$ vertices, $\theta(G) = l$. Then $G$ is not $(1, l-1)$-colorable iff $G \in \mathcal{B}(2, \frac{n}{2})$.*

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Suppose that $G$ is not $(1, l-1)$-colorable.

Then $l = \frac{n}{2}$, and so $n$ is even.

(If $l > \frac{n}{2}$, then there exists a clique $C_i$ s.t. $|C_i| = 1$, and so $G$ is $(1, l-1)$-colorable.)

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**Lemma (Epple, Ph.D. Thesis, 2011)**

*Let $G$ be a nonbipartite triangle free graph on $n$ vertices with $n$ even. Then $G$ is $(1, \frac{n}{2} - 1)$-colorable.*

**Lemma (Epple, Ph.D. Thesis, 2011)**

*Let $G$ be a bipartite graph on $n$ vertices with $n$ even. If $G$ is not a box cograph, then $G$ is $(1, \frac{n}{2} - 1)$-colorable.*

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$G$ is not $(1, l-1)$-colorable, $l = \frac{n}{2}$, $n$ is even. By Lemmas, $G$ is bipartite and is a box cograph, i.e., $G \in \mathcal{B}(2, \frac{n}{2})$. $\square$

## 5. Open problems

**Problem**

*Every graph $G$ with* $\omega(G) < 5 \Rightarrow \chi^b(G) \leq \theta(G) + 3$ *?*

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Remark: Every graph $G$ with $\omega(G) < 5 \Rightarrow \chi(G) \leq \theta(G) + 3$ ?

If this is true, then the problem is solved.

**Proof:** Let $G$ be a graph with $\omega(G) < 5$, $\theta(G) = l$, $l \geq 1$. Let $C_0 = \emptyset$, Let $C_1, C_2, \ldots, C_l$ be a partition of $V(G)$, s.t. each $C_j$ is a clique. For $i \in \{0, 1, \ldots, l-1\}$, let $G_i = G - \cup_{j=0}^{i} C_j$. **If** $\chi(G_i) \leq \theta(G_i) + 3 = l - i + 3$ **is true**. Then $V(G_i)$ can be partitioned into $l - i + 3$ independent sets $\{S_1, S_2, \ldots, S_{l-i+3}\}$. It follows that $\{S_1, S_2, \ldots, S_{l-i+3}, C_1, C_2, \ldots, C_i\}$ is an $(l - i + 3, i)$-coloring of $G$, i.e., $G$ is $(l - i + 3, i)$-colorable for each $i \in \{0, 1, \ldots, l-1\}$. Since $G$ is $(0, l)$-colorable, $G$ is $(t, l + 3 - t)$-color -able for each $t \in \{0, 1, 2, 3\}$. So $\chi^b(G) \leq \theta(G) + 3$.

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**Problem (Huang, GTCA(The 8th International Symposium on Graph Theory and Combinatorial Algorithms), 2019)**

*Characterize graphs $G$ for which $\chi^b(G) = \chi^c(G)$.*

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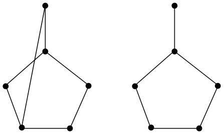
**Lemma (Ekim and Gimbel, Discrete Math. 2009)**

*The only triangle free graphs with $\chi(G) = \theta(G)$ are $P_3, K_1 \cup K_2, P_4, 2K_2, C_4, C_5$, together with the two graph are depicted below.*



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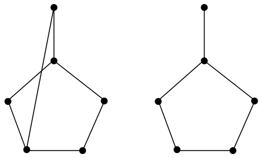
**Corollary**

*The only triangle free graphs with $\chi^b(G) = \chi^c(G)$ are $P_3, K_1 \cup K_2, P_4, C_5$, together with the two graph are depicted below.*



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**Problem (Huang, GTCA, 2019)**

*Characterize planar graphs $G$ for which $\chi^b(G) = \theta(G)$.*

*Thank you!*