Hypergraphs with infinitely many extremal constructions

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Definition

For an integer $r \geq 2$, an r-uniform hypergraph (henceforth r-graph) $\mathscr{H}$ is a collection of r-subsets of some finite set V.

- Given a family $\mathscr{F}$ of r-graphs we say $\mathscr{H}$ is **$\mathscr{F}$-free** if it does not contain any member of $\mathscr{F}$ as a subgraph.
- *Turán number* : The *Turán number* $\mathrm{ex}(n, \mathscr{F})$ of $\mathscr{F}$ is the maximum number of edges in an $\mathscr{F}$-free r-graph on n vertices.
- *Turán density* : The *Turán density* $\pi(\mathscr{F})$ of $\mathscr{F}$ is defined as

$$\pi(\mathscr{F}) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathscr{F})}{\binom{n}{r}}.$$

- *nondegenerate hypergraph* : A family $\mathscr{F}$ is called nondegenerate if $\pi(\mathscr{F}) > 0$.

Introduction

Theorem 1 (Mantel 1907)

$$\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem 2 (Turán 1941)

$$\mathrm{ex}(n, K_{\ell+1}) = |T(n, \ell)|,$$

where $T(n,\ell)$ is the balanced complete ℓ-partite graph on n vertices, i.e., Turán graph.

- W. Mantel, Solution to problem 28, by h. *Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh, and WA Wythoff, Wiskundige Opgaven* **10** (1907) 60–61.
- P. Turán, On an extermal problem in graph theory, *Mat. Fiz. Lapok* **48** (1941) 436–452.

Introduction

The **chromatic number** of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Theorem 3 (Erdős-Stone-Simonovits 1966)

Let H be a graph and $\chi(H) \geq 2$, then

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

- P. Erdős and A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946) 1087–1091.
- P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar* **1** (1966) 51–57.

Introduction

Corollary 4

Let H be a graph and $\chi(H) \geq 2$, then

$$\pi(H) = \frac{\chi(H)-2}{\chi(H)-1}.$$

Introduction

For $r \geq 3$ determining $\pi(\mathscr{F})$ for a family $\mathscr{F}$ of r-graphs is known to be notoriously hard in general.

Conjecture 5 (Turán 1941)

For every integer $\ell \geq 3$ we have $\pi(K^3_{\ell+1}) = 1 - 4/\ell^2$.

Erdős offered \$500 for the determination of any $\pi(K^r_\ell)$ with $\ell > r \geq 3$ and \$1000 for all $\pi(K^r_\ell)$ with $\ell > r \geq 3$.

P. Turán, On an extermal problem in graph theory, *Mat. Fiz. Lapok* **48** (1941) 436–452.

Introduction

Why Turán problem for hypergraph is so challenging ?

Theorem 6 (Kostochka 1982)

Assuming Turán's Tetrahedron conjecture is true, there are at least 2^{n-2} non-isomorphic extremal K_4^3-free constructions on $3n$ vertices.

Theorem 7 (Razborov 2010)

$$5/9 \leq \pi(K_4^3) \leq 0.561666.$$

A. V. Kostochka, A class of constructions for Turán's $(3,4)$-problem, *Combinatorica* **2** (1982) 187–192.

A. Razborov, On 3-hypergraphs with forbidden 4-vertex configurations, *SIAM Journal on Discrete Mathematics* **24** (2010) 946–963.

Stability

For a family $\mathscr{F}$ of r-graphs, it is natural to ask for the "continuity" of the discrete $\mathscr{F}$-free r-graphs whose size is close to $\mathrm{ex}(n, \mathscr{F})$.

- *stability* : Many families $\mathscr{F}$ have the property that there is a unique $\mathscr{F}$-free hypergraph $\mathscr{G}$ on n vertices achieving $\mathrm{ex}(n, \mathscr{F})$, and moreover, any $\mathscr{F}$-free hypergraph $\mathscr{H}$ of size close to $\mathrm{ex}(n, \mathscr{F})$ can be transformed to $\mathscr{G}$ by deleting and adding very few edges.

Introduction

Theorem 8 (Erdős-Simonovits 1968)

Fix $\ell \geq 2$. For every $\delta > 0$, there exists ε and $N_0 = N_0(\varepsilon)$ such that the following holds for every $n > N_0$ if G is an n-vertex graph containing no copy of $K_{\ell+1}$ with at least $(1-\varepsilon)|T(n,\ell)|$ edges, then G can be transformed to $T(n,\ell)$ by adding and deleting at most δn^2 edges.

M. Simonovits, A method for solving extremal problems in graph theory, stability problems, *Theory of Graphs (Proc. Colloq., Tihany, 1966)* (1968) 279–319

Non-stable

There are many Turán problems for hypergraphs that (perhaps) do not have the stability property.

- *non-stable* : For many families of r-uniform hypergraphs $\mathscr{M}$, there are perhaps many near-extremal $\mathscr{M}$-free configurations that are far from each other in edit-distance. Such a property is called non-stable.

Two famous examples:

- K_4^3
- Erdős-Sós Conjecture

Conjecture 9 (Erdős-Sós Conjecture)

Let $\mathscr{H}$ be a 3-graph with n vertices. If $L(v)$ is bipartite for all $v \in V(\mathscr{H})$, then $|\mathscr{H}| \leq (1/4 + o(1))\binom{n}{3}$.

If Erdős-Sós Conjecture is true, then it also does not have the stability property as there are several different near-extremal constructions.

Stability number

- *t-stable* : Let $r \geq 2$ and $t \geq 1$ be integers. A family $\mathscr{F}$ of r-graphs is t-stable if there exist m_0 and r-graphs $\mathscr{G}_1(m), \ldots, \mathscr{G}_t(m)$ on m vertices for every $m \geq m_0$ such that the following holds. For every $\delta > 0$ there exist $\varepsilon > 0$ and N_0 such that for all $n \geq N_0$ if $\mathscr{H}$ is an $\mathscr{F}$-free r-graph on n vertices with $|\mathscr{H}| > (1-\varepsilon)\mathrm{ex}(n, \mathscr{F})$ then $\mathscr{H}$ can be transformed to some $\mathscr{G}_i(n)$ by adding and removing at most δn^r edges.
- *stability number* : Denote by $\xi(\mathscr{F})$ the minimum integer t such that $\mathscr{F}$ is t-stable, and set $\xi(\mathscr{F}) = \infty$ if there is no such t. Call $\xi(\mathscr{F})$ *the stability number* of $\mathscr{F}$.

Introduction

Theorem 10 (Liu and Mubayi 2022)

If Conjecture 5 is ture i.e., $\pi(K_4^3) = 5/9$, *then* $\xi(K_4^3) = \infty$.

Theorem 11 (Liu and Mubayi 2022)

There exists a finite family $\mathscr{M}$ *of 3-graphs such that* $\xi(\mathscr{M}) = 2$.

This is the first finite 2-stable family of hypergraphs.

X. Liu and D. Mubayi, A hypergraph Turán problem with no stability, *Combinatorica* **42** (2022) 433–462.

Introduction

Theorem 12 (Liu, Mubayi and Reiher 2023+)

For every positive integer t there exists a finite family $\mathscr{M}$ of 3-graphs such that $\xi(\mathscr{M}) = t$.

Problem 13 (Liu and Mubayi 2022)

Determine $ex(n, \mathscr{F})$ for some family $\mathscr{F}$ with $\xi(\mathscr{F}) = \infty$.

Problem 14 (Liu, Mubayi and Reiher 2023+)

Does there exist a family $\mathscr{F}$ of triple systems with $\pi(\mathscr{F}) = 2/9$ but $\xi(\mathscr{F}) \neq 1$?

X. Liu and D. Mubayi, A hypergraph Turán problem with no stability, *Combinatorica* **42** (2022) 433–462.

X. Liu, D. Mubayi, and C. Reiher. Hypergraphs with many extremal configurations, *Israel. J. Math.* to appear.

Definition

- *blowup* : An r-graph $\mathscr{H}$ is a *blowup* of an r-graph $\mathscr{G}$ if there exists a map $\psi \colon V(\mathscr{H}) \to V(\mathscr{G})$ so that $\psi(E) \in \mathscr{G}$ iff $E \in \mathscr{H}$.
- *$\mathscr{G}$-colorable* : $\mathscr{H}$ is *$\mathscr{G}$-colorable* if there exists a map $\phi \colon V(\mathscr{H}) \to V(\mathscr{G})$ so that $\phi(E) \in \mathscr{G}$ for all $E \in \mathscr{H}$.

Main Result

Theorem 15 (Hou, Li, Liu, Mubayi and Zhang 2023+)

For every integer $t \geq 3$ there exists a finite family $\mathscr{F}_t$ of 3-graphs such that the following statements hold.

(1) *We have $\mathrm{ex}(n, \mathscr{F}_t) \leq \frac{(t-2)(t-1)}{6t^2} n^3$ for all $n \in \mathbb{N}$, and equality holds whenever $t \mid n$.*

(2) *If $t \mid n$, then the number of nonisomorphic maximum $\mathscr{F}_t$-free 3-graphs on n vertices is at least $n/2t$.*

(3) *We have $\xi(\mathscr{F}_t) = \infty$.*

(4) *For every integer $t \geq 4$ there exist constants $\varepsilon = \varepsilon(t) > 0$ and $N_0 = N_0(t)$ such that the following holds for every integer $n \geq N_0$. Every n-vertex $\mathscr{F}_t$-free 3-graph with minimum degree at least $\left(\frac{(t-2)(t-1)}{2t^2} - \varepsilon\right) n^2$ is Γ_t-colorable, where Γ_t is some fixed 3-graph on $t+2$ vertices.*

Multilinear polynomials

Denote by Δ_{m-1} the standard $(m-1)$-dimensional simplex, i.e.

$$\Delta_{m-1} = \{(x_1, \ldots, x_m) \in [0,1]^m : x_1 + \cdots + x_m = 1\}.$$

Given an m-variable continuous function f we define

$$\lambda(f) = \max\{f(x_1, \ldots, x_m) : (x_1, \ldots, x_m) \in \Delta_{m-1}\},$$

and

$$Z(f) = \{(x_1, \ldots, x_m) \in \Delta_{m-1} : f(x_1, \ldots, x_m) - \lambda(f) = 0\}.$$

Multilinear polynomials

We say p is *multilinear* if each term of p is of the form $\prod_{i \in S} x_i$ for some S.

We say p is *nonnegative* (or *nonpositive*) if $p(x_1, \ldots, x_m) \geq 0$ (or $p(x_1, \ldots, x_m) \leq 0$) for all $(x_1, \ldots, x_m) \in \Delta_{m-1}$. For a pair $\{i, j\} \subset [m]$ we say p is symmetric with respect to X_i and X_j if

$$p(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_m) = p(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_m).$$

Given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$ define the line segment $L(\vec{x}, \vec{y})$ with endpoints $\vec{x}$ and $\vec{y}$ as

$$L(\vec{x}, \vec{y}) = \{\alpha \cdot \vec{x} + (1 - \alpha) \cdot \vec{y} \colon \alpha \in [0, 1]\}.$$

Multilinear polynomials

Proposition 16

Let $p(X_1,\ldots,X_m) = p_1 + p_2(X_i + X_j) + p_3X_iX_j$ *be an* m*-variable multilinear polynomial that is symmetric with respect to* X_i *and* X_j*. Suppose that* p_3 *is nonnegative, and* p_4, p_5 *are nonnegative polynomials satisfying* $p_4 + p_5 = p_3$*. Then the* $(m+2)$*-variable polynomial*

$$\begin{aligned}&\hat{p}(X_1,\ldots,X_i,X_i',\ldots,X_j,X_j',\ldots,X_m)\\ &= p_1 + p_2(X_i + X_i' + X_j + X_j') + p_4(X_i + X_i')(X_j + X_j') + p_5(X_i + X_j)(X_i' + X_j')\end{aligned}$$

satisfies $\lambda(\hat{p}) = \lambda(p)$*, and moreover, for every* $(x_1,\ldots,x_m) \in Z(p)$ *we have* $L(\vec{y},\vec{z}) \subset Z(\hat{p})$*, where* $\vec{y},\vec{z} \in \Delta_{m+1}$ *are defined by*

$$\begin{aligned}\vec{y} &= \left(x_1,\ldots,x_{i-1},(x_i+x_j)/2,0,x_{i+1},\ldots,x_{j-1},0,(x_i+x_j)/2,x_{j+1},\ldots,x_m\right),\\ \vec{z} &= \left(x_1,\ldots,x_{i-1},0,(x_i+x_j)/2,x_{i+1},\ldots,x_{j-1},(x_i+x_j)/2,0,x_{j+1},\ldots,x_m\right).\end{aligned}$$

Lagrangian

For an r-graph $\mathscr{G}$ on m vertices, the multilinear polynomial $p_{\mathscr{G}}$ is defined by

$$p_{\mathscr{G}}(X_1,\ldots,X_m) = \sum_{E\in\mathscr{G}} \prod_{i\in E} X_i.$$

The *Lagrangian* of $\mathscr{G}$ is defined by $\lambda(\mathscr{G}) = \lambda(p_{\mathscr{G}})$. Define

$$Z(\mathscr{G}) = Z(p_{\mathscr{G}}) = \left\{(x_1,\ldots,x_n) \in \Delta_{m-1} : p_{\mathscr{G}}(x_1,\ldots,x_m) = \lambda(\mathscr{G})\right\}.$$

Crossed blowup

Definition 17 (Crossed blowup)

Let $\mathcal{G}$ be a 3-graph and $\{v_1, v_2\} \subset \mathcal{G}$ be a pair of vertices with $d(v_1, v_2) = k \geq 2$. Fix an ordering of the vertices in $N_{\mathcal{G}}(v_1, v_2)$, say $N_{\mathcal{G}}(v_1, v_2) = \{u_1, \ldots, u_k\}$. The crossed blowup $\mathcal{G} \boxplus \{v_1, v_2\}$ of $\mathcal{G}$ on $\{v_1, v_2\}$ is defined as follows.

(1) *Remove all edges containing the pair $\{v_1, v_2\}$ from $\mathcal{G}$,*

(2) *add two new vertices v_1' and v_2', make v_1' a clone of v_1 and v_2' a clone of v_2,*

(3) *for every $i \in [k-1]$ add the edge set $\{u_i v_1 v_1', u_i v_1 v_2', u_i v_2 v_1', u_i v_2 v_2'\}$, and for $i = k$ add the edge set $\{u_k v_1 v_2, u_k v_1 v_2', u_k v_1' v_2, u_k v_1' v_2'\}$.*

Crossed blowup

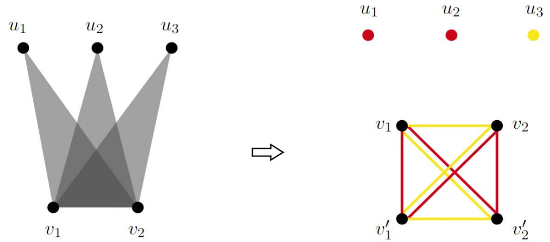

Figure 1: $\{u_1v_1v_2, u_2v_1v_2, u_3v_1v_2\}$ and $\{u_1v_1v_2, u_2v_1v_2, u_3v_1v_2\} \boxplus \{v_1, v_2\}$. The link of red vertices is the red $K_{2,2}$, the link of the yellow vertex is the yellow $K_{2,2}$.

Crossed blowup

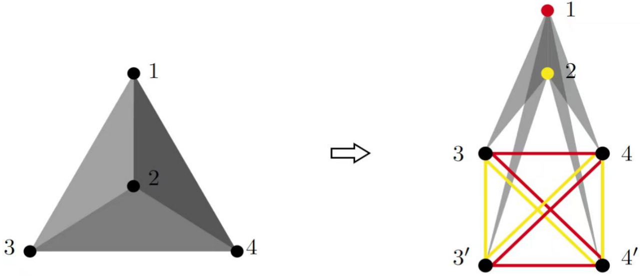

Figure 2: K_4^3 and $K_4^3 \boxplus \{3,4\}$.

Crossed blowup

Let $\mathscr{G}$ be a 3-graph. A pair $\{v_1, v_2\} \subset V(\mathscr{G})$ is symmetric in $\mathscr{G}$ if

$$L_{\mathscr{G}}(v_1) - v_2 = L_{\mathscr{G}}(v_2) - v_1.$$

The crossed blowup of a 3-graph has the following properties.

Proposition 18

Suppose that $\mathscr{G}$ is an m-vertex 3-graph and $\{v_1, v_2\} \subset V(\mathscr{G})$ is a pair of vertices with $d(v_1, v_2) \geq 2$. Then the following statements hold.

(1) *The 3-graph $\mathscr{G}$ is contained in $\mathscr{G} \boxplus \{v_1, v_2\}$ as an induced subgraph. In particular, $\lambda(\mathscr{G}) \leq \lambda(\mathscr{G} \boxplus \{v_1, v_2\})$.*

(2) *The 3-graph $\mathscr{G} \boxplus \{v_1, v_2\}$ is 2-covered iff $\mathscr{G}$ is 2-covered.*

(3) *If $\{v_1, v_2\}$ is symmetric in $\mathscr{G}$, then $\lambda(\mathscr{G} \boxplus \{v_1, v_2\}) = \lambda(\mathscr{G})$. If, in addition, there exists $(x_1, \ldots, x_m) \in Z(\mathscr{G})$ with $x_1 + x_2 > 0$, then the set $Z(\mathscr{G} \boxplus \{v_1, v_2\})$ contains a one-dimensional simplex (i.e. a nontrivial line segment).*

Main definition

Definition 19

Let $t \geq 1$ be an integer.

(1) *Let*

$$\Gamma_{t+2} = \begin{cases} \{134, 234\} \boxplus \{3,4\} & \text{if } t = 1, \\ K_{t+2}^{3} \boxplus \{t+1, t+2\} & \text{if } t \geq 2. \end{cases}$$

(2) *Let $\mathfrak{d}_{t+2}$ be the collection of all Γ_{t+2}-colorable 3-graphs.*

(3) *Let $\gamma_{t+2}(n) = \max\{|\mathscr{H}| \colon v(\mathscr{H}) = n$ and $\mathscr{H} \in \mathfrak{d}_{t+2}\}$.*

(4) *Let $\mathscr{F}_{t+2} = \{F \colon v(F) \leq 4(t+4)^2$ and $F \notin \mathfrak{d}_{t+2}\}$.*

Blowup-invariance

Given an r-graph F we say $\mathscr{H}$ is *F-hom-free* if there is no homomorphism from F to $\mathscr{H}$. This is equivalent to say that every blowup of $\mathscr{H}$ is F-free. For a family $\mathscr{F}$ of r-graphs we say $\mathscr{H}$ is *$\mathscr{F}$-hom-free* if it is F-hom-free for all $F \in \mathscr{F}$. An easy observation is that if an r-graph F is 2-covered, then $\mathscr{H}$ is F-free iff it is F-hom-free.

Definition 20 (Blowup-invariance)

A family $\mathscr{F}$ of r-graphs is blowup-invariant if every $\mathscr{F}$-free r-graph is also $\mathscr{F}$-hom-free.

Turán number

For every r-graph $\mathscr{G}$ let $\mathscr{F}_{\infty}(\mathscr{G})$ be the (infinite) family of all r-graphs that are not $\mathscr{G}$-colorable, i.e.

$$\mathscr{F}_{\infty}(\mathscr{G}) = \{r\text{-graph } F\colon \text{ and } F \text{ is not } \mathscr{G}\text{-colorable}\}.$$

For every positive integer M define the family $\mathscr{F}_M(\mathscr{G})$ of r-graphs as

$$\mathscr{F}_M(\mathscr{G}) = \{F \in \mathscr{F}_{\infty}(\mathscr{G})\colon v(F) \le M\}.$$

Lemma 21

For every r-graph $\mathscr{G}$ and every positive integer M the family $\mathscr{F}_M(\mathscr{G})$ is blowup-invariant.

Turán number

Let $\mathscr{H}$ be an r-graph and $\{u,v\} \subset V(\mathscr{H})$ be two non-adjacent vertices (i.e., no edge contains both u and v). We say u and v are *equivalent* if $L_{\mathscr{H}}(u) = L_{\mathscr{H}}(v)$ (in particular, two equivalent vertices are non-adjacent). Otherwise we say they are *non-equivalent*. An equivalence class of $\mathscr{H}$ is a maximal vertex set in which every pair of vertices are equivalent. We say $\mathscr{H}$ is *symmetrized* if it does not contain non-equivalent pairs of vertices.

Theorem 22

Suppose that $\mathscr{F}$ is a blowup-invariant family of r-graphs. If $\mathfrak{H}$ denotes the class of all symmetrized $\mathscr{F}$-free r-graphs, then $ex(n,\mathscr{F}) = \mathfrak{h}(n)$ holds for every $n \in \mathbb{N}^+$, where $\mathfrak{h}(n) = \max\{|\mathscr{H}| \colon \mathscr{H} \in \mathfrak{H}$ and $v(\mathscr{H}) = n\}$.

Vertex-extendibility

Definition 23 (Vertex-extendibility)

Let $\mathscr{F}$ be a family of r-graphs and let $\mathfrak{H}$ be a class of $\mathscr{F}$-free r-graphs. We say that $\mathscr{F}$ is vertex-extendable with respect to $\mathfrak{H}$ if there exist $\zeta > 0$ and $N_0 \in \mathbb{N}$ such that for every $\mathscr{F}$-free r-graph $\mathscr{H}$ on $n \geq N_0$ vertices satisfying $\delta(\mathscr{H}) \geq (\pi(\mathscr{F})/(r-1)! - \zeta)n^{r-1}$ the following holds: if $\mathscr{H} - v$ is a subgraph of a member of $\mathfrak{H}$ for some vertex $v \in V(\mathscr{H})$, then $\mathscr{H}$ is a subgraph of a member of $\mathfrak{H}$ as well.

Degree stability

Theorem 24 (Liu, Mubayi and Reiher 2023)

Suppose that $\mathscr{F}$ is a blowup-invariant nondegenerate family of r-graphs and that $\mathfrak{H}$ is a hereditary class of $\mathscr{F}$-free r-graphs. If $\mathfrak{H}$ contains all symmetrized $\mathscr{F}$-free r-graphs and $\mathscr{F}$ is vertex-extendable with respect to $\mathfrak{H}$, then the following statement holds. There exist $\varepsilon > 0$ and N_0 such that every $\mathscr{F}$-free r-graph on $n \geq N_0$ vertices with minimum degree at least $(\pi(\mathscr{F})/(r-1)! - \varepsilon)n^{r-1}$ is contained in $\mathfrak{H}$.

X. Liu, D. Mubayi, and C. Reiher. A unified approach to hypergraph stability, *J. Combin. Theory Ser. B*, 158:36C62, 2023

Feasible region

- *shadow* : The *shadow* of $\mathscr{H}$ is defined as

$$\partial\mathscr{H} = \left\{ A \in \binom{V(\mathscr{H})}{r-1} : \text{there is } B \in \mathscr{H} \text{ such that } A \subset B \right\}.$$

- *density* : The *edge density* of $\mathscr{H}$ is defined as $\rho(\mathscr{H}) = |\mathscr{H}|/\binom{v(\mathscr{H})}{r}$, and the *shadow density* of $\mathscr{H}$ is defined as $\rho(\partial\mathscr{H}) = |\partial\mathscr{H}|/\binom{v(\mathscr{H})}{r-1}$.
- *feasible region* : For a family $\mathscr{F}$ the *feasible region* $\Omega(\mathscr{F})$ of $\mathscr{F}$ is the set of points $(x, y) \in [0, 1]^2$ such that there exists a sequence of $\mathscr{F}$-free r-graphs $(\mathscr{H}_k)_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} v(\mathscr{H}_k) = \infty, \quad \lim_{k\to\infty} \rho(\partial\mathscr{H}_k) = x, \quad \text{and} \quad \lim_{k\to\infty} \rho(\mathscr{H}_k) = y.$$

Feasible region

- $\text{proj}\Omega(\mathscr{F})\colon = \{x\colon \text{there is } y \in [0,1] \text{ such that } (x,y) \in \Omega(\mathscr{F})\}$.
- *feasible region function* : The function $g(\mathscr{F})\colon \text{proj}\Omega(\mathscr{F}) \to [0,1]$ such that

$$\Omega(\mathscr{F}) = \left\{(x,y) \in [0, c(\mathscr{F})] \times [0,1]\colon 0 \le y \le g(\mathscr{F})(x)\right\}.$$

Theorem 25 (Liu and Mubayi 2021)

The feasible region function $g(\mathscr{F})$ is not necessarily continuous. But $g(\mathscr{F})$ is a left-continuous almost everywhere differentiable function.

X. Liu and D. Mubayi, The feasible region of hypergraphs, *Journal of Combinatorial Theory, Series B* **148** (2021) 23–59.

Feasible region

Problem 26 (Liu, Mubayi and Reiher 2023+)

For $r \geq 3$ does there exist a non-degenerate family $\mathscr{F}$ of r-graphs so that $g(\mathscr{F})$ has infinitely many global maxima? If so, can the set $M(\mathscr{F})$ be uncountable? Can it even contain a non-trivial interval?

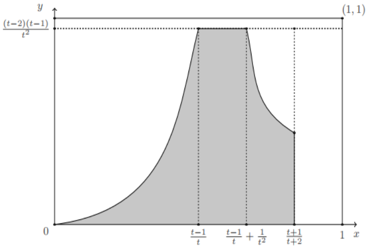

Figure 4: The function $g(\mathcal{F}_t)$ attains its maximum on the interval $\left[\frac{t-1}{t}, \frac{t-1}{t} + \frac{1}{t^2}\right]$.

Thank You for Your Attention!