Uniqueness and Rapid Mixing in the Bipartite Hardcore Model

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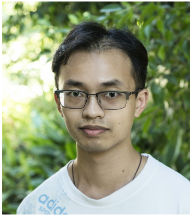
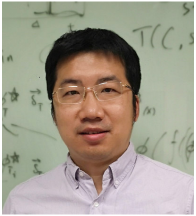
based on joint work with



Jingcheng Liu



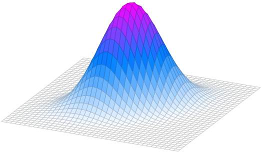
Yitong Yin

Sampling problem:

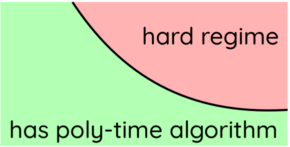
Draw (approximate) random samples from a distribution

Gibbs distribuiton:

- high-dimensional joint distribution
- described by few parameters and local interactions
- typical input size: poly(n)



$\{0, 1\}^n$



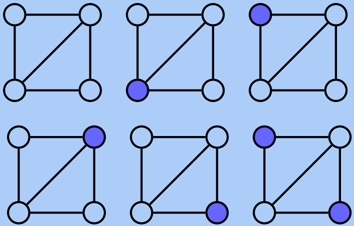
Computational phase transition:

computational complexity of sampling problem changes sharply around certain parameter values

Counting & sampling independent set (#IS)

- $G = ([n], E)$ with n vertices and max degree Δ.
- #IS: how many independent sets are there in G?

An example



number of independent set: 6

- exact counting independent set is #P-hard.
- approximate counting $\overset{\text{[JVV86]}}{\Longleftrightarrow}$ approximate sampling:
  - $\Delta \leq 5$, poly-time algorithm for approx. sampling [Wei06]
  - $\Delta \geq 6$, no poly-time algorithm unless **NP = RP** [Sly10]

Hardcore model

- $G = ([n], E)$ with n vertices and max degree Δ.
- Fugacity $\lambda > 0$ is a real number.
- $\mathrm{Ind}(G) = \{S \subseteq [n] \mid S$ is an independent set$\}$.
- Gibbs distribution

$$\forall S \in \mathrm{Ind}(G), \quad \mu(S) := \frac{\lambda^{|S|}}{Z}, \quad \text{where } Z_G(\lambda) := \sum_{I \in \mathrm{Ind}(G)} \lambda^{|I|}.$$

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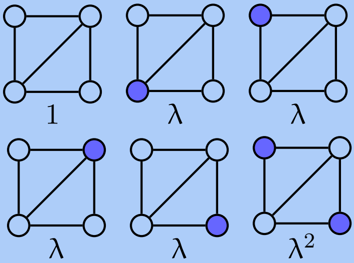


Partition function:

$$Z = 1 + 4\lambda + \lambda^2$$

This model is self-reducible

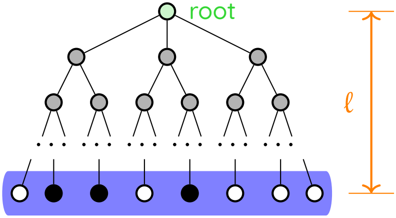
Computational phase transition

On Δ-regular tree:

uniqueness $\lambda_c(\Delta)$ non-uniqueness

$\lambda : 0 \longrightarrow \infty$

tree uniqueness threshold: $\lambda_c(\Delta) := (\Delta - 1)^{(\Delta-1)}/(\Delta - 2)^{\Delta} \approx \frac{e}{\Delta}$



Tree uniqueness

$\mathbf{Pr}_{S\sim\mu}$ [root $\in S \mid \sigma$] does not depend on σ when $\ell \to \infty$

if and only if $\lambda \le \lambda_c(\Delta)$

σ : boundary condition on level ℓ

On general graph with maximum degree Δ:

easy $\lambda_c(\Delta)$ hard

$\lambda : 0 \longrightarrow \infty$

Computational phase transition:

- $\lambda < \lambda_c$: poly-time algorithm for approx. sampling [Wei06]
- $\lambda > \lambda_c$: no poly-time algorithm unless **NP = RP** [Sly10]

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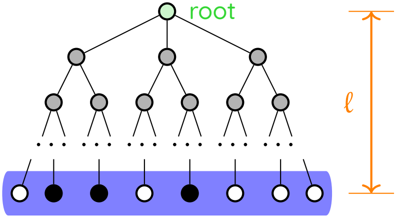

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Hardcore model on bipartite graph (weighted #BIS)

It is easy: there is a poly-time algorithm to find a matximum independent set in the bipartite graph (Kőnig's theorem[1]).

It is hard: many important problems are proved to be #BIS-equivalent or #BIS-hard under AP-reductions.

Selected examples

- stable matchings (counting)
- ferro. Potts model (parti. func.)
- ferro. Ising with mixed external fields (parti. func.)

[DGGJ04, GJ07, DGJ10, CGM12 DGJR12, GJ12a, BDG+13, LLZ14, GJ15, CGG+16, GŠVY16, GGY21]

Conjecture[DGGJ04]:
#BIS represents an intermediate complexity class:

- it has no FPRAS in general
- it is easier than #SAT

[1] In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

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Previous algorithmic results

Non-uniqueness regime:

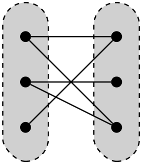
- α-expander bipartite graph:
  - $\lambda \geq (C_0\Delta)^{4/\alpha}$, an $n^{O(\log \Delta)}$ time sampler [JKP20]
  - $\lambda \geq (C_1\Delta)^{6/\alpha}$, an $O(n \log n)$ time sampler [CGG+21]
  - $\lambda \geq (C_2\Delta)^{2/\alpha}$, an $n^{O(\log \Delta)}$ time sampler [FGKP23]
- Δ-regular α-expander bipartite graph:
  - $\lambda \geq \frac{f(\alpha)\log \Delta}{\Delta^{1/4}}$, an $n^{O(\Delta)}$ time sampler [JPP22]
- random Δ-regular bipartite graph:
  - $\Delta \geq \Delta_0, \lambda \geq \frac{\log^4 \Delta}{\Delta}$, an $n^{O(1)}$ time sampler [LLLM19]
  - $\Delta \geq \Delta_1, \lambda \geq \frac{50 \log^2 \Delta}{\Delta}$, an $n^{1+O(\frac{\log^2(\Delta)}{\Delta})}$ time sampler [JKP20]
  - $\Delta \geq \Delta_2, \lambda \geq \frac{100 \log \Delta}{\Delta}$, an $O(n \log n)$ time sampler [CGŠV22]
- unbalanced bipartite graph:
  - $6\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $n^{O(\log(\Delta_L\Delta_R))}$ time sampler [CP20]
  - $3.4\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $n^{O(\log(\Delta_L\Delta_R))}$ time sampler [FGKP23]
  - $(1+e)\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $O(n \log n)$ time sampler [BCP22]

Previous algorithmic results

Uniqueness regime: [. . ., AJKPV22, CFYZ22, CE22]

- general graph: if $\lambda < \lambda_c(\Delta)$, there is an $O(n \log n)$ time sampler
- bipartite graph: if $\lambda = 1, \Delta_L \leq 5$, an $O(n^2)$ time sampler [LL15]
  ($\lambda = 1 \wedge \lambda < \lambda_c(\Delta) \Leftrightarrow \Delta \leq 5$)

$$\lambda_c(\Delta) = \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^{\Delta}}$$



Our results

For $\delta \in (0, 1)$, $\Delta_L \geq 3$, if $\lambda \leq (1 - \delta)\lambda_c(\Delta_L)$, then

- the system is in the uniqueness regime
- there is a sampler that runs in time

$$T := n \left(\frac{\Delta_L \log n}{\lambda} \right)^{O(1/\delta)}$$

- the **mixing time** of **Glauber dynamics** is bounded by $O(n^2) \cdot T$

- When $\Delta_L = 1$, G is a forest, which is trivial.
- When $\Delta_L = 2$, this model becomes an Ising model. Our results still work, but since $\lambda_c(2) = \infty$, it is quite technical to state them here.

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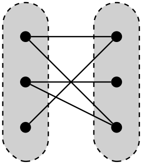

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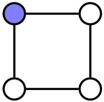
Glauber dynamics for Hardcore model:

start from an arbitrary independent set X_0;
for t from 1 to T **do**:

- pick a vertex $v \in V$ uniformly at random;
- with prob. $\frac{\lambda}{1+\lambda}$, let $S = X_{t-1} \cup \{v\}$;
  with prob. $\frac{1}{1+\lambda}$, let $S = X_{t-1} \setminus \{v\}$;
- **if** $S \in \mathrm{Ind}(G)$ **then** $X_t = S$ **else** $X_t = X_{t-1}$;



irreducible + aperiodic + reversible $\Longrightarrow X_t \sim \mu$ as $t \to \infty$

mixing time: essential running time of Glauber dynamics

$$T_{\mathrm{mix}} := \max_{X_0} \min\{t \mid D_{\mathrm{TV}}(X_t \parallel \mu) \le 1/100\}$$

total variation distance: conanical distance between distributions

$$D_{\mathrm{TV}}(X_t \parallel \mu) := \frac{1}{2} \sum_{S \in \mathrm{Ind}(G)} \left| \mathbf{Pr}\left[X_t = S\right] - \mu(S) \right|$$

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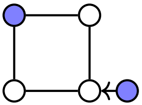

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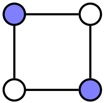

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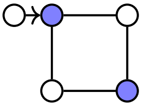

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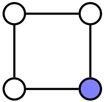

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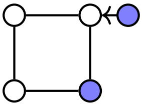

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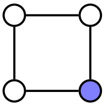

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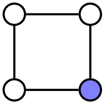

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Background

Proof outline

Fast sampler

Mixing of Glauber dynamics on $\mathbb{L} \cup \mathbb{R}$

Background

Let ν be a distribution over $\Omega = \{-1, +1\}^n$. $\forall \sigma \in \Omega, \quad \|\sigma\|_+ = |\{i \mid \sigma_i = 1\}|$

impose external field $\theta > 0$

$\theta * \nu$: a distribution on Ω:

$$\forall \sigma, \quad (\theta * \nu)(\sigma) \propto \nu(\sigma) \cdot \theta^{\|\sigma\|_+}$$

flip the distribution

$\overline{\nu}$: a distribution on Ω:

$$\forall \sigma, \quad \overline{\nu}(\sigma) = \nu(-\sigma)$$

- hardcore model: μ (fugacity λ) $\Longrightarrow \theta * \mu$ (fugacity $\theta\lambda$)

An example

μ: hardcore model with fugacity λ

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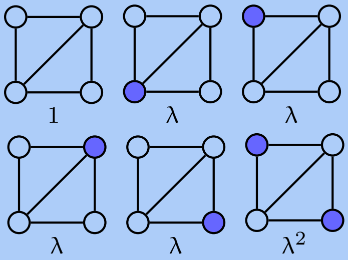

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$$\forall \sigma, \quad (\theta * \nu)(\sigma) \propto \nu(\sigma) \cdot \theta^{\|\sigma\|_+}$$

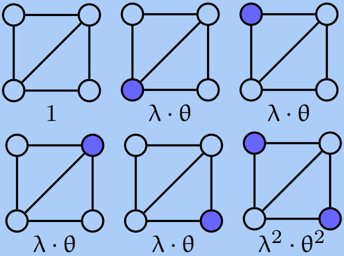
flip the distribution

$\overline{\nu}$: a distribution on Ω:

$$\forall \sigma, \quad \overline{\nu}(\sigma) = \nu(-\sigma)$$

- hardcore model: μ (fugacity λ) $\Longrightarrow$ $\theta * \mu$ (fugacity $\theta\lambda$)

An example



$\theta * \mu$: hardcore model with fugacity $\theta\lambda$

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For $0 < \theta$, Field dynamics $P^{FD}_{\theta,\nu}$: Markov chain $(X_t)_{t \geq 0}$ on Ω:

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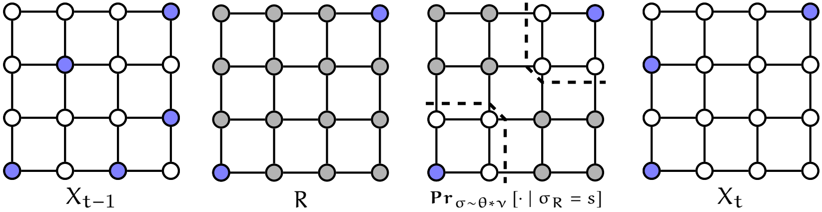

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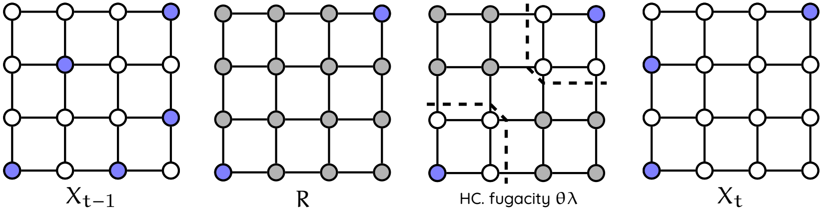

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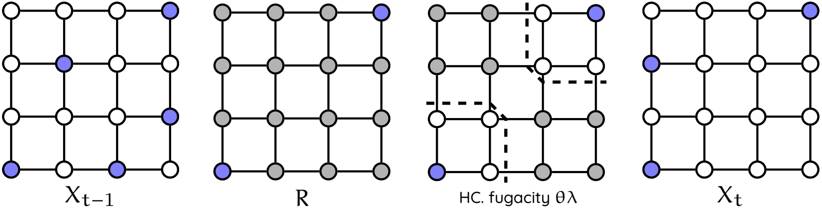

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Let ν be a distribution over $\{-1,+1\}^n$ and $X \sim \nu$ be a random vector.

influence matrix $\Psi_\nu \in \mathbb{R}^{n\times n}$ [ALO20]

$$\Psi_\nu(i,j) := \mathbf{Pr}\left[X_j = +1 \mid X_i = +1\right] - \mathbf{Pr}\left[X_j = +1 \mid X_i = -1\right] = \frac{\mathrm{Cov}(X_i,X_j)}{\mathrm{Var}(X_i)}$$

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in self-reducible models

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Ψ_ν is similar to $\mathrm{Corr}(X)$

K-marginal stability [CFYZ22, CE22]

there is $\rho \in \{\nu, \overline{\nu}\}$ that for $i \in [n]$, $S \subseteq \Lambda \subseteq [n] \setminus \{i\}$, $\tau \in \Omega(\rho_\Lambda)$,

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Proof outline

Fast sampler

Mixing of Glauber dynamics on $L \cup R$

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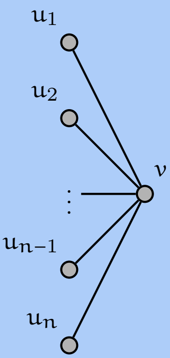
example



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- $\|\Psi_\mu\|_\infty = \frac{n}{2}$

What could we do? 🤔

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Maybe we could take $\nu = \mu_L$.

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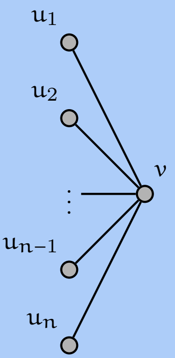

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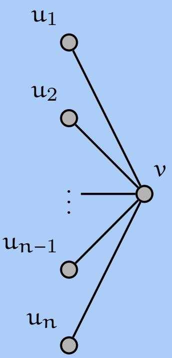

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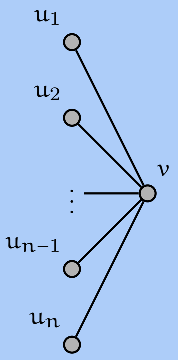

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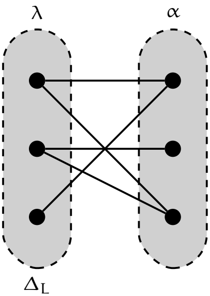

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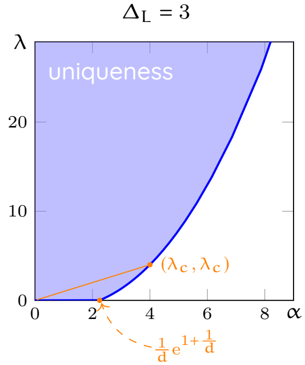

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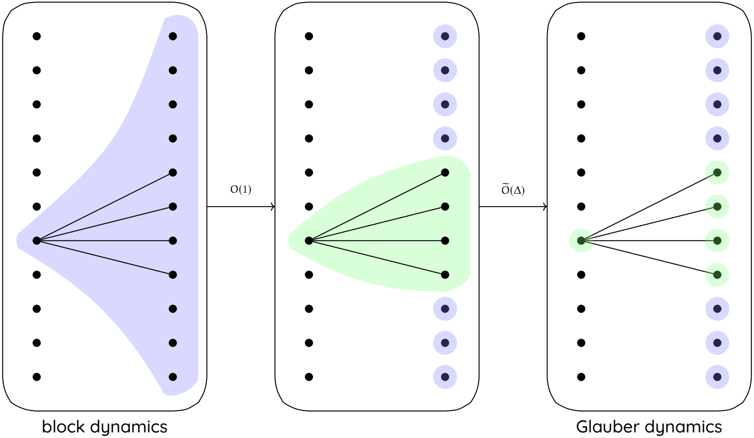
Proof outline: mixing of GD on μ

- Glauber dynamics on $\nu = \mu_L$ is rapidly mixing.
- It works like a block dynamics that update a random vertex on the left and all the vertices on the right in each step.
- We finish the proof by comparing it and the Glauber dynamics on μ via the **block factorization** [CMT15, CP20, CLV21].



Thank you

arXiv:2305.00186

Summary

For $\delta \in (0,1)$, $\Delta_L \geq 3$, if $\lambda \leq (1-\delta)\lambda_c(\Delta_L)$, then

- the system is in the uniqueness regime
- there is a sampler that runs in time
$$T := n\left(\frac{\Delta_L \log n}{\lambda}\right)^{O(1/\delta)}$$
- the mixing time of Glauber dynamics is bounded by $O(n^2) \cdot T$

Open problems

- Remove the depedency on Δ_L in the running time of the sampler.
- Better mixing time for the Glauber dynamics.
- Bipartite hardcore model for negative λ.