Permanents and permutation statistics

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Determinants and permanents

For a matrix $A = [a_{i,j}]_{1 \leq i,j \leq n}$ over a commutative ring with identity, its determinant and permanent are defined by

$$\det(A) = \det[a_{i,j}]_{1 \leq i,j \leq n} = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

and

$$\mathrm{per}(A) = \mathrm{per}[a_{i,j}]_{1 \leq i,j \leq n} = \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i,\pi(i)}$$

respectively, where $\mathfrak{S}_n$ is the symmetric group of all permutations of $[n] := \{1, \dots, n\}$.

$$\mathrm{per} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 3$$

Determinants v.s. Permanents

Permanents are useful in Combinatorics!

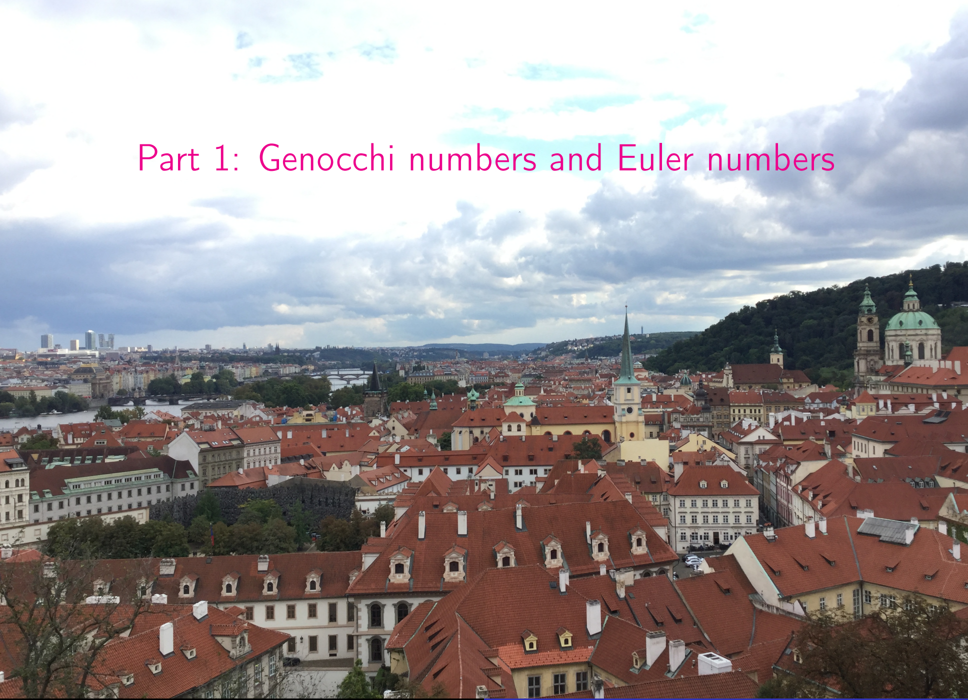
- (Valiant 1979) The complexity of computing the permanent of $n \times n$ $(0,1)$-matrices is NP-hard.
- The permanent of the biadjacency matrix of a bipartite graph G counts the perfect matchings.
- **Permanent-Determinant Method**: If A' is a coherent signing of biadjacency matrix A, then $\mathrm{per}(A) = |\det(A')|$.

L.G. Valiant, The complexity of computing the permanent, Theoretical Comput. Sci., **8** (1979), 189–201.

V.V. Vazirani and M. Yannakakis, Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs, Discrete Appl. Math., 25 (1989), 179–190.

G. Kuperberg, Symmetries of Plane Partitions and the Permanent-Determinant method, J. Combin. Theory Ser. A 68 (1994), 115–151.

Part 1: Genocchi numbers and Euler numbers



Sun's permanent conjecture

The Genocchi numbers (of the first kind) G_n are defined by

$$\frac{2x}{e^x+1} = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}.$$

Conjecture (Zhi-Wei Sun, Question 403386 on MathOverflow)

$$\mathrm{per}\left[\left\lfloor \frac{2j-k}{n} \right\rfloor\right]_{1\le j,k\le n} = -G_{n+1}.$$

$$\mathrm{per}\begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \mathrm{per}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 3$$

Kreweras' triangle

$$\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}, \begin{bmatrix}1&1&0&0&0\\1&1&1&1&0\\1&1&1&1&1\\0&1&1&1&1\\0&0&0&1&1\end{bmatrix}, \begin{bmatrix}1&1&0&0&0&0&0\\1&1&1&1&0&0&0\\1&1&1&1&1&1&0\\1&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&0&0&1&1&1&1\\0&0&0&0&0&1&1\end{bmatrix}$$

- Dumont 1974: $\#\mathcal{D}_{2n+1} = (-1)^n G_{2n}$ with

  $\mathcal{D}_{2n+1} := \{\sigma \in \mathfrak{S}_{2n+1} : \forall\, i \in [2n],\ \sigma(i) > \sigma(i+1) \text{ iff } \sigma(i) \text{ is even}\}.$
- Kreweras' triangle $K_{2n-1,k} = \#\{\sigma \in \mathcal{D}_{2n+1} : \sigma(1) = k+1\}$
- Four interpretations of Kreweras' triangle via rearrangement and Foata's first fundamental transformation

Genocchi numbers of the second kind (or median Genocchi numbers) $H_{2n-1} = (-1)^n \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} G_{2n-2j}$

Conjecture (P. Luschny, A005439 on OEIS)

$$\mathrm{per} \left[\left\lfloor \frac{2j-k-1}{2n} \right\rfloor \right]_{1 \le j,k \le 2n} = (-1)^n H_{2n-1}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Proof: In bijection with Dellac configurations known to be counted by the normalized median Genocchi numbers

Euler numbers

Euler number E_n counts up-down permutations in $\mathfrak{S}_n$

Conjecture (D. Chen, Question 402572 at MathOverflow)

Let $P_n := [\operatorname{sgn}(\sin \pi \frac{i+j}{n+1})]_{1\le i,j\le n}$. *Then*

$$\operatorname{per}(P_{2n}) = \operatorname{per}(P_{2n}^{-1}) = (-1)^n E_{2n}.$$

Conjecture (D. Chen, Question 403336 at MathOverflow)

Let $Q_n := [\operatorname{sgn}(\sin \pi \frac{i+2j}{n+1})]_{1\le i,j\le n}$. *Then* $\operatorname{per}(Q_n) = (-1)^n E_n$.

Conjecture (D. Chen, Question 402572 at MathOverflow)

Let $A_{2n} := \left[\operatorname{sgn}\left(\tan \pi \frac{i+j}{2n+1}\right)\right]_{1\le i,j\le 2n}$. *Then*

$$\operatorname{per}(A_{2n}) = \operatorname{per}(A_{2n}^{-1}) = (-1)^n E_{2n}.$$

Signed Eulerian identities

$$P_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Roselle's signed Eulerian identities:

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{exc}(\pi)} = \begin{cases} 0 & \text{if } n = 2m+1, \\ (-1)^m E_{2m} & \text{if } n = 2m, \end{cases}$$

where

$$\mathrm{exc}(\pi) := |\{i \in [n] : \pi(i) > i\}|,$$
$$\mathfrak{D}_n := \{\pi \in \mathfrak{S}_n : \pi(i) \neq i \text{ for all } i \in [n]\}.$$

An action on matrices

Actions on all -1's in P_{2n}:

$$P_4^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}, \quad P_6^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

1st Way Consider an exc-variant denoted as exc_P, and show that both exc and exc_P have the same sign-balance over derangements.

2nd Way Show that $\mathrm{per}(P_{2n}) = \mathrm{per}(P_{2n}^{-1})$ via some elementary operation on matrices.

Definition (An action on matrix)

Define $\phi_{k,\ell}(A)$ to be the matrix obtained from A by multiplying the k-th row and the ℓ-th column by -1. Then $\mathrm{per}(\phi_{k,\ell}(A)) = \mathrm{per}(A)$.

$\mathrm{per}(A_{2n}) = \mathrm{per}(A_{2n}^{-1})$

$$A_4 = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 0 \\ 1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

Actions on all -1's of A_{2n}:

$$A_4^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}, \quad A_6^{-1} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 & -1 & -1 \end{bmatrix}.$$

An intriguing bijection

$$A_6^{-1} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 & -1 & -1 \end{bmatrix} \Rightarrow \pi = 315624, \ \mathrm{exph}(\pi) = 3.$$

Theorem

$$\sum_{\pi \in \mathfrak{S}_{2n}} t^{\mathrm{exc}(\pi)} y^{\mathrm{fix}(\pi)} = \sum_{\pi \in \mathfrak{S}_{2n}} t^{\mathrm{exph}(\pi)} y^{\mathrm{fix}(\pi)}.$$

Replace each i with $\phi(i)$ in the two-line notation of $\pi \in \mathfrak{S}_{2n}$, where

$$\phi(i) = \begin{cases} n+k & \text{if } i = 2k-1 \text{ for some } 1 \leq k \leq n, \\ k & \text{if } i = 2k \text{ for some } 1 \leq k \leq n. \end{cases}$$

The group of the action $\phi_{k,l}$

Definition

For any $S \subseteq [n] \times [n]$, define the transformation matrix T_S with respect to S by

$$\Big(\prod_{(k,l)\in S} \phi_{k,l}\Big)(A) = A \circ T_S,$$

where A is any $n \times n$ matrix over $\mathbb{R}$ and $\circ$ is the Hadamard product.

Let us consider the set of transformation matrices

$$\mathcal{T}_n := \{ T_S : S \subseteq [n] \times [n] \}.$$

For instance, $\mathcal{T}_2$ consists of four matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

The group of the action $\phi_{k,l}$

Proposition

For any positive integer n, the transformation group $\mathcal{T}_n$ is isomorphic to the group $\mathbb{Z}_2^{2n-2}$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Consequently, $|\mathcal{T}_n| = 2^{2n-2}$.

For any subset $S \subseteq [n] \times [n]$, we define $\phi_S := \prod_{(k,l)\in S} \phi_{k,l}$, and call S a kernel if ϕ_S is the identity action. If S is a kernel, then for any $(a, b) \in S$ we have $\phi_{(a,b)} = \phi_{S\setminus\{(a,b)\}}$.

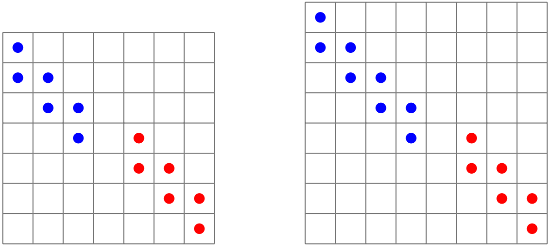


Figure: Examples of generators for $\mathcal{T}_n$ when $n = 7, 8$

Part 2: Binomial transformation of Euler numbers

Binomial transformation of Euler numbers

Conjecture (D. Chen, Question 404768 at MathOverflow)

Let $R_n := [\mathrm{sgn}(\cos\pi\frac{i+j}{n+1})]_{1\le i,j\le n}$*. Then*

$$\mathrm{per}(R_n) = \begin{cases} -\sum_{k=0}^{m}\binom{m}{k}E_{2k+1} & \text{if } n = 2m+1, \\ \sum_{k=0}^{m}\binom{m}{k}E_{2k} & \text{if } n = 2m. \end{cases}$$

$$R_4 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, R_5 = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

Step 1: adjust R_n

$$\tilde{R}_1 = \begin{bmatrix} \color{red}{1} \end{bmatrix}, \ \tilde{R}_3 = \begin{bmatrix} 1 & 1 & 0 \\ \color{red}{1} & \color{red}{1} & \color{red}{1} \\ 0 & -1 & -1 \end{bmatrix}, \ \tilde{R}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 \\ \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix},$$

$$\tilde{R}_2 = \begin{bmatrix} 1 & 1 \\ \color{red}{1} & \color{red}{1} \end{bmatrix}, \ \tilde{R}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ 1 & -1 & -1 & -1 \end{bmatrix}, \ \tilde{R}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Definition (A variant of excedances)

For $\pi \in \mathfrak{S}_n$, define

$$\widetilde{\mathrm{exc}}(\pi) := |\{i \in [n] : \pi(i) > i \text{ and } \pi(i) \neq \lceil (n+1)/2 \rceil\}|.$$

Note that $\mathrm{exc}(\pi) - \widetilde{\mathrm{exc}}(\pi)$ equals 1 or 0 depending on whether $\lceil \frac{n+1}{2} \rceil$ is an excedance top or not.

Step 2: interprete permanent and explain cancellation

$$\mathrm{per}(\tilde{R}_{2m}) = \sum_{\pi\in\mathfrak{S}_{2m}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} \quad \text{and} \quad \mathrm{per}(\tilde{R}_{2m+1}) = \sum_{\pi\in\tilde{\mathfrak{D}}_{2m+1}} (-1)^{\widetilde{\mathrm{exc}}(\pi)},$$

where $\tilde{\mathfrak{D}}_{2m+1} = \{\pi \in \mathfrak{S}_{2m+1} : \mathrm{Fix}(\pi) \subseteq \{m+1\}\}$.

Lemma

Via Foata's first fundamental transformation and Foata–Strehl action:

$$\sum_{\pi\in\mathfrak{S}_{2m}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = \sum_{\pi\in\mathfrak{S}^*_{2m}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = (-1)^{m-1}|\mathfrak{S}^*_{2m}|,$$

$$\sum_{\pi\in\tilde{\mathfrak{D}}_{2m+1}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = \sum_{\pi\in\mathfrak{S}^*_{2m+1}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = (-1)^{m}|\mathfrak{S}^*_{2m+1}|.$$

Foata's first fundamental transformation

Foata's "transformation fondamentale" $o : \pi \mapsto o(\pi)$:

- Write π^{-1} in *standard cycle form*:
  1. each cycle has its largest letter in the leftmost position;
  2. the cycles are listed from left to right in increasing order of their largest letters.
- The one-line notation of $o(\pi)$ is obtained from the standard cycle form of π^{-1} by erasing all the parentheses.

Lemma

Foata's first fundamental transformation $o : \mathfrak{S}_n \to \mathfrak{S}_n$ satisfies $\mathrm{Exct}(\pi) = \mathrm{Dest}(o(\pi))$ for each $\pi \in \mathfrak{S}_n$.

$$\mathrm{Exct}(\pi) := \{\pi(i) : i < \pi(i)\}$$
$$\mathrm{Dest}(\pi) = \{\pi(i) : \pi(i) > \pi(i+1)\}$$

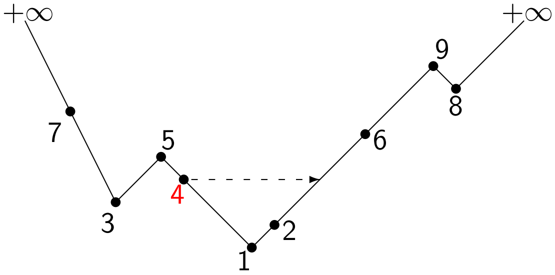


Figure: The Foata–Strehl action φ_x on 735412698 with $x = 4$.

D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, Math. Z. **137** (1974), 257–264.

Step 3: Poupard numbers via alternating permutations

Definition (Foata–Han 2013)

For any $\pi \in \mathfrak{S}_n$ with $\pi(i) = n$, define

$$\mathrm{grn}(\pi) := \max\{\pi(i-1), \pi(i+1)\},$$

called the **greater neighbour** of n in π. Let $\mathfrak{A}_n$ denote the set of alternating (down-up) permutations and let

$$\mathfrak{A}_{n,k} := \{\pi \in \mathfrak{A}_n : \mathrm{grn}(\pi) = k\}$$

Lemma

For each $n \geq 1$, there exists a bijection $f : \mathfrak{S}_n^* \to \mathfrak{A}_{n+1, \lfloor \frac{n+1}{2} \rfloor}$.

Foata–Han's generating function formulae

Theorem (Foata–Han 2013)

Denote $g_n(k) := |\mathfrak{A}_{2n-1,k-1}|$ *and* $h_n(k) := |\mathfrak{A}_{2n,k}|$. *Then*

$$1+\sum_{n\geq 1}\sum_{1\leq k\leq 2n+1} g_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\frac{y^{k-1}}{(k-1)!}=\sec(x+y)\cos(x-y),$$

$$1+\sum_{n\geq 1}\sum_{1\leq k\leq 2n+1} h_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\frac{y^{k-1}}{(k-1)!}=\sec^2(x+y)\cos(x-y).$$

Lemma

Central Poupard numbers:

$$g_{n+1}(n+1)=\sum_{k=0}^{n}\binom{n}{k}E_{2k} \quad \textit{and} \quad h_{n+1}(n+1)=\sum_{k=0}^{n}\binom{n}{k}E_{2k+1}.$$

Bala's continued fraction conjecture

Conjecture (P. Bala, A005799 on OEIS)

The number $2^{-n}\sum_{k=0}^{n}\binom{n}{k}E_{2k}$ *has the exponential generating function formula*

$$\sum_{n\geq 0}\frac{2^{-n}\sum_{k=0}^{n}\binom{n}{k}E_{2k}}{n!}z^n=\cfrac{2}{2-\cfrac{1-e^{-4z}}{2-\cfrac{1-e^{-8z}}{2-\cfrac{1-e^{-12z}}{\cdots}}}}.$$

Descent polynomials on the multiset $\{1,1,\ldots,n,n\}$

Theorem

Let $\mathfrak{S}_n^{(2)}$ be the set of multipermutations on $\{1,1,\ldots,n,n\}$. Then

$$\sum_{\pi\in\mathfrak{S}_{2n}} t^{\widetilde{\mathrm{exc}}(\pi)} = 2^n \sum_{\pi\in\mathfrak{S}_n^{(2)}} t^{\mathrm{des}(\pi)}.$$

The polynomial $\sum_{\pi\in\mathfrak{S}_n^{(2)}} t^{\mathrm{des}(\pi)}$ that we denote $A_n^{(2)}(t)$ is called the nth 2-Eulerian polynomial. F. Ardila (2020) proved that 2-Eulerian polynomials are the h-polynomials of the dual bipermutahedron.

Theorem (MacMahon, Combinatory analysis)

MacMahon's factorial generating function formula for $A_n^{(2)}(t)$

$$\frac{A_n^{(2)}(t)}{(1-t)^{2n+1}} = \sum_{k\geq 0} \binom{k+2}{2}^n t^k.$$

Exponential generating function for $A_n^{(2)}(t)$

Via Touchard's continued fraction formula:

$$\sum_{k\geq 0} q^{\binom{k+1}{2}} z^k = \cfrac{1}{1-z+\cfrac{(1-q)z}{1-z+\cfrac{(1-q^2)z}{1-z+\cfrac{(1-q^3)z}{\cdots}}}}.$$

Theorem (Setting $t=-1$ implies Bala's conjecture)

The e.g.f. for $A_n^{(2)}(t)$ has continued fraction expansion

$$\sum_{n\geq 0} \frac{tA_n^{(2)}(t)}{n!} z^n = t-1+\cfrac{1-t}{1-t+\cfrac{(1-e^{(1-t)^2 z})t}{1-t+\cfrac{(1-e^{2(1-t)^2 z})t}{1-t+\cfrac{(1-e^{3(1-t)^2 z})t}{\cdots}}}}.$$

Part 3: Combinatorics of the γ-positivity

Palindromic and unimodal

A polynomial $h(t) = \sum_{i=0}^{d} h_i t^i \in \mathcal{R}[t]$ is

- palindromic if $h_i = h_{d-i}$ for all i
- unimodal if for some c

$$h_0 \leq h_1 \leq \cdots \leq h_c \geq \cdots \geq h_{d-1} \geq h_d$$

Example 1: $h(t) = 1 + 20t + 48t^2 + 20t^3 + t^4$
Example 2: $h(t) = (1 + t)^n$

The 1989 survey of Stanley:
Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in Graph Theory and Its Applications: East and West

The 2014 survey of Brändén:
Unimodality, log-concavity, real-rootedness and beyond, in Handbook of Enumerative Combinatorics

γ-positivity

Definition (γ-positive)

$h(t)$ is palindromic $\Longleftrightarrow$ it can be expanded as

$$h(t) = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k t^k (1+t)^{d-2k}.$$

If $\gamma_k \geq 0$ for all k, then $h(t)$ is said to be γ-positive.

- Example: $h(t) = 1 + 20t + 48t^2 + 20t^3 + t^4$ is γ-positive, as

$$\begin{aligned} h(t) &= 1(1+t)^4 + 16t + 42t^2 + 16t^3 \\ &= 1(1+t)^4 + 16t(1+t)^2 + 10t^2 \end{aligned}$$

- γ-positive $\Longrightarrow$ palindromic and unimodal (why?)

The 2018 survey of Athanasiadis

Gamma-positivity in combinatorics and geometry, SLC77

The Eulerian polynomials are γ-positive

Double descent of π: $\pi_{i-1} > \pi_i > \pi_{i+1}$
NDD_n: set of all permutations in $\mathfrak{S}_n$ with no double descents

Theorem (Foata & Schützenberger 1970)

The Eulerian polynomials are γ-positive:

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = \#\{\pi \in \mathrm{NDD}_n : \mathrm{des}(\pi) = k, \pi_1 < \pi_2\}$.

Many proofs are known: recurrence, Foata–Strehl action, **cd**-index, analysis (real-rootedness), continued fractions, symmetric functions, poset topology (Rees products), ...

Multiset Eulerian polynomials

The 2-Eulerian polynomial:

$$A_n^{(2)}(t) = \sum_{\pi} t^{\mathrm{des}(\pi)}$$

summed over all permutations π of $\{1^2, 2^2, \ldots, n^2\}$.

- Ardila (2022) $\mathcal{B}_n$ is bipermutohedron: $h_{\mathcal{B}_n^*}(t) = A_n^{(2)}(t)$
- Carlitz and Hoggatt (1978) proved that $A_n^{(2)}(t)$ is palindromic
- Simion (1984) proved that $A_n^{(2)}(t)$ is real-rooted

Since a palindromic polynomial with only real roots is γ-positive, it is natural to ask:

Problem

Is there any combinatorial interpretation for the γ-coefficients $\gamma_{n,k}^{(2)}$ of $A_n^{(2)}(t)$?

Answer: weakly increasing trees & permanent of $\left[\mathrm{sgn}(\cos\pi\frac{i+j}{n+1})\right]$

Weakly increasing trees on a multiset

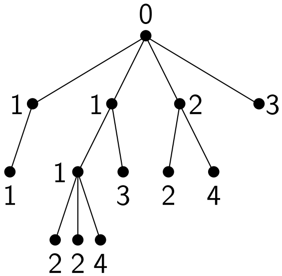


Figure: A weakly increasing tree on $M = \{1^4, 2^4, 3^2, 4^2\}$.

Weakly increasing trees on a multiset

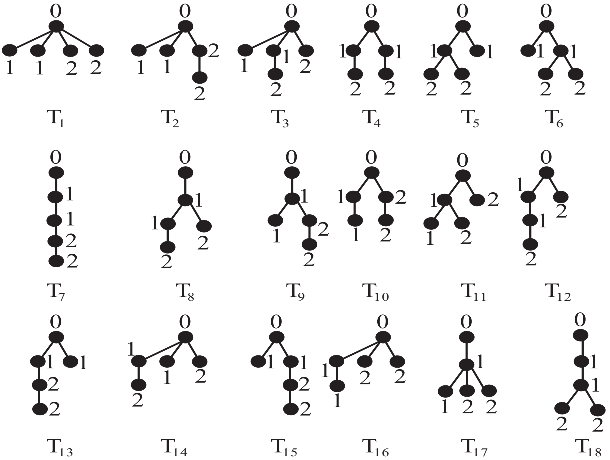


Figure: All 18 weakly increasing trees on the multiset $\{1^2, 2^2\}$

- Gessel (1990) proved that the number of alternating multipermutations $\pi \in \mathfrak{S}_n^{(2)}$:

$$\pi(1) \le \pi(2) > \pi(3) \le \pi(4) > \pi(5) \le \cdots,$$

is $2^{-n} \sum_{k=0}^{n} \binom{n}{k} E_{2k}$.

- Lin–Ma–Ma–Zhou (2021) interpreted the γ-coefficients $\gamma_{n,k}^{(2)}$ in

$$A_n^{(2)}(t) = \sum_{k=0}^{n-1} \gamma_{n,k}^{(2)} t^k (1+t)^{2(n-1)-2k}$$

as some class of weakly increasing trees.

Corollary

The number of alternating multipermutations in $\mathfrak{S}_n^{(2)}$ equals the number of weakly increasing trees on $\{1, 1, \ldots, n-1, n-1, n\}$ with n leaves and without young leaves.

Combinatorics of the γ-positivity of $\tilde{A}_{2m}(t)$

By Foata's first fundamental transformation:

$$2^n A_n^{(2)}(t) = \tilde{A}_{2m}(t) := \sum_{\pi \in \mathfrak{S}_{2m}} t^{\widetilde{\mathrm{exc}}(\pi)} = \sum_{\pi \in \mathfrak{S}_{2m}} t^{\widetilde{\mathrm{des}}(\pi)},$$

where $\widetilde{\mathrm{des}}(\pi) := |\mathrm{Dest}(\pi) \setminus \{m+1\}|$.

Theorem (Provides a new interpretation for $\gamma_{n,k}^{(2)}$)

Denote by $\mathrm{Val}(\pi)$ the set of valleys of π. Then $\tilde{A}_{2m}(t)$ has the γ-positivity expansion

$$\tilde{A}_{2m}(t) = \sum_{\pi \in \mathfrak{S}_{2m}} t^{\widetilde{\mathrm{des}}(\pi)} = \sum_{k=0}^{m-1} |\tilde{D}_{2m,k}| t^k (1+t)^{2m-2-2k},$$

where $\tilde{D}_{2m,k}$ is the set of $\pi \in \mathfrak{S}_{2m}$ with $\mathrm{Ddes}(\pi) \setminus \{m+1\} = \emptyset$, $m+1 \notin \mathrm{Val}(\pi)$ and $\widetilde{\mathrm{des}}(\pi) = k$.

Step 1: an intriguing equidistribution

Lemma

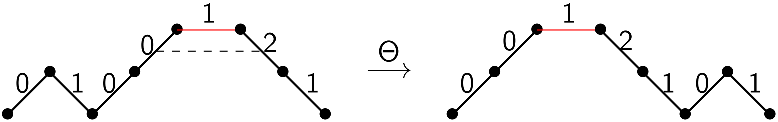
There exists a bijection η preserving the number of descents between

$$P_m := \{\pi \in \mathfrak{S}_{2m} : \mathrm{Ddes}(\pi) = \emptyset, m+1 \textit{ is a peak}\}$$

and

$$V_m := \{\pi \in \mathfrak{S}_{2m} : \mathrm{Ddes}(\pi) = \emptyset, m+1 \textit{ is a valley}\}.$$

Via the Françon–Viennot bijection that encodes permutations as Laguerre histories and involution Θ below:



For $x \in [2m]$ and $\pi \in \mathfrak{S}_{2m}$, introduce the restricted Foata–Strehl action

$$\widetilde{\varphi}_x(\pi) = \begin{cases} \varphi'_x(\pi), & \text{if } x \neq m+1; \\ \pi, & \text{if } x = m+1. \end{cases}$$

If we denote $\hat{\pi}$ the unique permutation in $\mathrm{Orb}(\pi)$ with $\mathrm{Ddes}(\hat{\pi}) \setminus \{m+1\} = \emptyset$, then

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\widetilde{\mathrm{des}}(\sigma)} = t^{\widetilde{\mathrm{des}}(\hat{\pi})}(1+t)^{|\mathrm{Dasc}(\hat{\pi})\setminus\{m+1\}|}.$$

- If $m+1$ is a double descent or a double ascent of $\hat{\pi}$, then

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\widetilde{\mathrm{des}}(\sigma)} = t^{\widetilde{\mathrm{des}}(\hat{\pi})}(1+t)^{2m-2-2\widetilde{\mathrm{des}}(\hat{\pi})}.$$

- If $m+1$ is a peak of $\hat{\pi}$, then $m+1$ is a valley of $\eta(\hat{\pi})$ and

$$\sum_{\sigma \in \mathrm{Orb}(\pi) \biguplus \mathrm{Orb}(\eta(\hat{\pi}))} t^{\widetilde{\mathrm{des}}(\sigma)} = t^{\widetilde{\mathrm{des}}(\hat{\pi})}(1+t)^{2m-2-2\widetilde{\mathrm{des}}(\hat{\pi})}.$$

Part 4: Perfect matchings in bipartite graphs

证明思路:(晏卫根 & 赵彤远合作)

二部图

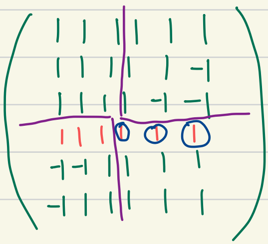
应用完美匹配积分公式:

$$P(G,x)=\sum_{r=0}^{n}(-1)^r p(G,r)\,x^{n-r}$$

Then $\int_0^{+\infty}P(G,x)e^{-x}dx = \#$ perfect matchings in $\bar{G}$

↓ 二部补图

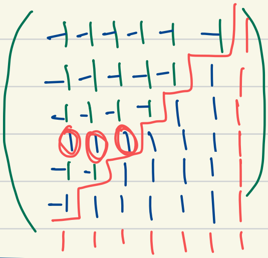
应用于矩阵



得到积分式

$$\sum_{i,j}(-2)^{2n-i-j}S(n,i)S(n,j)(i+j)!$$

应用于矢阶阵



得到积分式为

$$\sum_{k,j}2^{2n-k-1}(-1)^{n-k}\binom{n-1}{j}(k+1)!\,S(2n-1-j,k)$$

另一方面,应用取 $\frac{\cos(x-y)}{\cos(x+y)}$ 对角系数方式

① $\left[\frac{x^ny^n}{(n!)^2}\right]\cos(x-y)\cdot\sec(x+y)=\sum_{k\geq 0}\binom{n}{k}E_{2k}$

② $\left[\frac{x^ny^n}{(n!)^2}\right]\frac{\cos(x-y)}{\cos(x+y)}$ ← $\frac{e^{i\theta}+e^{-i\theta}}{2}=\cos\theta$

$=\left[\frac{x^ny^n}{(n!)^2}\right]\frac{e^{i(x-y)}+e^{-i(x-y)}}{e^{i(x+y)}+e^{-i(x+y)}}$ (用Euler公式 $e^{i\theta}=\cos\theta+i\sin\theta$)

$=\left[\frac{x^ny^n}{(n!)^2}\right]\frac{e^{2ix}+e^{2iy}}{1+e^{2i(x+y)}}$

$=\left[\frac{x^ny^n}{(n!)^2}\right]\frac{\frac{1}{2}(e^{2ix}+e^{2iy})}{1+(-1)(e^{2i(x+y)}-1)}$

应用 $\sum_{n,k}k!S(n,k)y^k\frac{x^n}{n!}=\frac{1}{1-y(e^x-1)}$

$=\sum_{k,j}2^{2n-k}(-1)^{n-k}\binom{n}{j}k!\,S(2n-j,k)$

二项式递归 + Stirling 递归

The type B Stirling numbers $S_B(n,k)$ have the e.g.f:

$$\sum_{n\geq 0} S_B(n,k)\frac{x^n}{n!} = \frac{1}{2^k k!} e^x (e^{2x}-1)^k.$$

The tangent numbers E_{2k+1} have the e.g.f.

$$\sum_{n\geq 0} \frac{E_{2n+1}\, x^{2n+1}}{(2n+1)!} = \tan x$$

定理:

$$\sum_{i,j} (-1)^{i+j+1} (i+j+1)!\, S_B(n,i) S_B(n+1,j) = \sum_k \binom{n}{k} E_{2k+1}$$

定理:

$$\sum_{i,j} (-1)^{i+j} (i+j)!\, S_B(n,i) S_B(n,j) = E_{2n}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\left.\begin{array}{l} \frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta \\ \frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta \end{array}\right\} \Rightarrow \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$$

$$\tan'\theta = \sec^2\theta = \frac{4}{(e^{i\theta}+e^{-i\theta})^2}$$

Thank you for your attention