On the polynomial reconstruction of digraphs and graphs

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Vertex Reconstruction Conjecture of Graphs

- The famous Ulam's conjecture states that each simple graph $G = (V, E)$ with $n \geq 3$ vertices be uniquely reconstructed from its deck $\{G - v | v \in V(G)\}$. Each $G - v$ is called a card of G.

$$\text{Obviously,}\ \ G \Rightarrow \{G - v | v \in V\}.$$

$$\text{Ulam's Conjecture}:\ \ \{G - v | v \in V\} \Rightarrow \text{unique } G \text{ if n} \geq 3?$$

- According to reliable sources (Kelly's doctoral thesis appeared in 1942), it was discovered in Wisconsin in 1941 by Kelly and Ulam .

S. M. Ulam, *A Collection of Mathematical Problems*, Wiley (Interscience), New York, 1960, p29.

J. A. Bondy, R. L. Hemminger, *Graph Reconstruction—Survey*, Journal of Graph Theory, 1(1977), 227-268.

- The trees are reconstructible (Manvel, Canad. J. Math., 1970)).
- The maximal outerplanar graphs are reconstructible (Manvel, DM, 1972).
- Regular graphs are reconstructible (The proof is easy).
- The unicyclic graphs are reconstructible (Arjomandi, Corneil, Canad. J. Math., 1974).
- The disconnected graphs are reconstructible (Manvel, JCTB, 1976).

- McKay verified by computer that the conjecture holds for $3 \leq V(G) \leq 10$. (McKay, JGT, 1977).
- Godsil and McKay (JCTB, 1981) proved that a graph is reconstructible if all but at most one of its eigenvalues are simple and have eigenvectors not orthogonal to the vector j with all entries equal to one.
- Hong (JCTB, 1982) proved that if there exists a card $G - v$ of G none of whose eigenvectors is orthogonal the vector j with all entries equal to one, then G is reconstructible from its deck.
- Bollobás (JGT, 1990) showed that almost every graph has reconstruction number three, a conjecture by Harary and Plantholt (JGT, 1985).
- Kostochka, Nahvi, West and Zirlin (Eur. J Combin., 2021) proved that 3-regular graphs are 2-reconstructible.

- Wang obtained the following interesting result: Suppose that A and B are two integral symmetric matrices such that $\det(xI-A)=\det(xI-B)$ and $\det(xI-A_i)=\det(xI-B_i)$ for each i. If $\det(xI-A)$ is irreducible over $\mathcal{Q}[x]$, then there exists a diagonal matrix D with each diagonal entry being ± 1 such that $B=D^TAD$.
- Although some reconstructible graphs are obtained, the Ulam's conjecture is still open.

Wei Wang, *A uniqueness theorem on matrices and reconstruction*, Journal of Combinatorial Theory, Ser. B, 99 (2009), 261–265.

List of some recent references on the graph reconstruction

- S. K. Gupta, P. Mangal, V. Paliwal, *Some work towards the proof of the reconstruction conjecture*, Discrete Mathematics, 272(2003), 291–296.
- R. Forman, *Finite-type invariants for graphs and graph reconstructions*, Advances in Mathematics, 186 (2004), 181–228.
- Wei Wang, *A uniqueness theorem on matrices and reconstruction*, Journal of Combinatorial Theory, Ser. B, 99 (2009), 261–265.
- A. V. Kostochka, M. Nahvi, D. B. West and D. Zirlin, *3-regular graphs are 2-reconstructible*, European Journal of Combinatorics, 91(2021), 103216.
- T. Hosaka, *The reconstruction conjecture for finite simple graphs and associated directed graphs*, Discrete Mathematics, 345 (2022), 112893.

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- Harary posed a similar conjecture (i.e., the edge reconstruction conjecture), which states that every simple graph G with edge set $E(G)$ can be reconstructed from its edge deck $\{G-e|e \in E(G)\}$ for $|E(G)| \geq 4$.

$$\text{Harary's Conjecture}: \{G-e|e \in E(G)\} \Rightarrow G \text{ if } |E(G)| \geq 4?$$

- Lovász (JCTB, 1972) proved that the edge reconstruction conjecture holds for simple graphs with n vertices and at least $\frac{n(n-1)}{4}$ edges.

$$\text{If } |E(G)| \geq n(n-1)/4, \text{then } \{G-e|e \in E(G)\} \Rightarrow G.$$

F. Harary, *On the reconstruction of a graph from a collection of subgraphs*, Theory of Graphs and Its Applications (M. Fiedler, ed.), Czechoslovak Academy of Sciences, Prague/Academic Press, New York, 1965, 47–52.

Edge Reconstruction Conjecture of Graphs

- Müller (JCTB, 1977) improved the Lovász's result and proved that the edge reconstruction conjecture holds for simple graphs with n vertices and more than $n \cdot \log_2 n$ edges.

$$\text{If } |E(G)| \geq n \log_2 n, \text{then } \{G - e | e \in E(G)\} \Rightarrow G.$$

- Godsil, Krasikov and Roditty (JCTB, 1987) proved that if $2^{m-k} > n!$ or if $2m > \binom{n}{2} + k$, then G is reconstructible from its collection of k-edge deleted subgraphs, where $m = |E(G)|$.

- Although Müller's result implies that the edge reconstruction conjecture holds for almost all simple graphs, the Harary's conjecture is still open.

- If G is reconstructible and has no isolated vertices, then G is edge reconstructible, i.e.,

$$Ulam's\ Conj.\ holds \ \Rightarrow \ Harary's\ Conj.\ holds.$$

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- Tutte proved that the characteristic polynomial $f(G;x)$ of a graph G can be reconstructed from the deck $\{G-v|v\in V\}$.

$$\{G-v|v\in V\}\Rightarrow f(G;x).$$

- Tutte proved that if the characteristic polynomial $f(G;x)$ of a graph G is irreducible over the rationals, then G is reconstructible.

$$\{G-v|v\in V\}\Rightarrow G \text{ if } f(G;x) \text{ is irreducible over } \mathcal{Q}.$$

W. T. Tutte, All the king's horses, in: Graph Theory and Related Topics, edited by J. A. Bondy and U. S. R. Murty (Academic Press, New York), 1979, 15–33.

- Cvetković, at the XVIII International Scientific Colloquium in Ilmenau in 1973, posed a related problem as follows: Can the characteristic polynomial $f(G;x)$ of a simple graph G with vertex set $V(G)$ be reconstructed from $\{f(G-v;x)|v \in V(G)\}$ for $|V(G)| \geq 3$?

$$\{f(G-v;x)|v \in V(G)\} \Rightarrow f(G;x) \text{ if } |V(G)| \geq 3?$$

- The same problem was independently posed by Schwenk. Gutman and Cvetković obtained some results related to this problem.

A. J. Schwenk, *Spectral reconstruction problems*, Ann. New York Acad. Sci., 328 (1979), 183–189.

I. Gutman, D. Cvetković, *The reconstruction problem for characteristic polynomials of graphs*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498 (451) (1975), 45–48.

- Hagos proved that $f(G;x)$ is reconstructible from $\{f(G-v;x)|v \in V(G)\} \cup \{f((G-v)^c;x)|v \in V(G)\}$.

$$\{f(G-v;x)|v \in V(G)\} \cup \{f((G-v)^c;x)|v \in V(G)\} \Rightarrow f(G;x).$$

- Note that $f'(G;x) = \sum_{v \in V(G)} f(G-v;x)$. Hence the coefficients of $f(G;x)$ except for the constant term can be reconstructed from $\{f(G-v;x)|v \in V(G)\}$.

- No examples of non-unique reconstruction of the characteristic polynomial of graphs are known.

E. M. Hagos, *The characteristic polynomial of a graph is reconstructible from the characteristic polynomials of its vertex-deleted subgraphs and their complements*, the electronic journal of combinatorics, 7 (2000), #R12

- Under the assumption that the reconstruction of the characteristic polynomial is not unique, Cvetković described some properties of graphs G such that the constant term of $f(G;x)$ can not be reconstructed from $\{f(G-v;x) | v \in V(G)\}$.

- Sciriha and Stanić survey classical and some more recent results concerning the reconstruction problem of the characteristic polynomial of graphs.

D. Cvetković, *On the reconstruction of the characteristic polynomial of a graph*, Discrete Mathematics, 212 (2000), 45–52.

I. Sciriha and Z. Stanić, *The polynomial reconstruction problem: The first 50 years*, Discrete Mathematics, 346 (2023), 113349.

- **A natural problem is: Can the characteristic polynomial $f(G;x)$ (resp. permanental polynomial $g(G;x)$) of a graph or digraph be reconstructed from $\{f(G-e;x)|e\in E(G)\}$ (resp. $\{g(G-e;x)|e\in E(G)\}$)?**

$$\{f(G-e;x)|e\in E(G)\} \Rightarrow f(G;x) \text{ if } |E(G)|\geq 4?$$

$$\{g(G-e;x)|e\in E(G)\} \Rightarrow g(G;x) \text{ if } |E(G)|\geq 4?$$

Outline

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- Let $X = (x_{st})_{n\times n}$ be a matrix of order n over the complex field. For any $1 \leq i, j \leq n$, define a matrix $X_{ij} = (x_{st}^{ij})_{n\times n}$, where

$$x_{st}^{ij} = \begin{cases} x_{st} & \text{if } (s,t) \neq (i,j) \\ 0 & \text{if } (s,t) = (i,j). \end{cases}$$

That is, X_{ij} is the matrix obtained from matrix X by replacing the (i,j)-entry x_{ij} with 0. For example, if

$$X = \begin{pmatrix} a & b & c \\ e & f & g \\ r & s & t \end{pmatrix},$$

then

$$X_{12} = \begin{pmatrix} a & 0 & c \\ e & f & g \\ r & s & t \end{pmatrix}, X_{33} = \begin{pmatrix} a & b & c \\ e & f & g \\ r & s & 0 \end{pmatrix}.$$

Obviously, if $x_{ij} = 0$, then $X = X_{ij}$.

- Let $\varphi : S_n \to \mathbb{R}$ be an function, where S_n is the symmetric group of order n.

- The φ-immanant of X is defined as

$$\mathrm{Imm}_{\varphi}(X) = \sum_{\alpha \in S_n} \varphi(\alpha) x_{1\alpha(1)} x_{2\alpha(2)} \cdots x_{n\alpha(n)},$$

  where the sum ranges over all elements α of S_n.

- Obviously, $\mathrm{Imm}_{\varphi}(X) = \det(X)$ if $\varphi(\alpha) = sgn(\alpha)$ for any $\alpha \in S_n$ and $\mathrm{Imm}_{\varphi}(X) = \mathrm{per}(X)$ if $\varphi(\beta) = 1$ for any $\beta \in S_n$.

Theorem 1 (Zhang, Jin, Yan, 2023)

Let $X = (x_{st})_{n\times n}$ be a matrix of order n over the complex field and let X_{ij} be defined as above. Then for any function $\varphi : S_n \to \mathbb{R}$, the φ-immanant of X satisfies:

$$(n^2 - n)\mathrm{Imm}_{\varphi}(X) = \sum_{1\le i,j\le n} \mathrm{Imm}_{\varphi}(X_{ij}). \tag{1}$$

The following theorem is immediate from Theorem 1.

Theorem 2 (Zhang, Jin, Yan, 2023)

Let $X = (x_{st})_{n\times n}$ be a matrix of order n over the complex field and let X_{ij} be defined as above. Then the determinant of X satisfies:

$$(n^2 - n)\det(X) = \sum_{1\le i,j\le n} \det(X_{ij}). \tag{2}$$

Theorem 3 (Zhang, Jin, Yan, 2023)

Let $X = (x_{st})_{n\times n}$ be a matrix of order n over the complex field and let m be the number of non-zero entries of X. Then

$$(m - n)\det(X) = \sum_{(i,j)\in I} \det(X_{ij}), \tag{3}$$

where $I = \{(i,j)|x_{ij} \neq 0, 1 \le i,j \le n\}$.

- By the definition of the determinant and permanent,

$$\det(X) = \sum_{\alpha \in S_n} sgn(\alpha) x_{1,\alpha(1)} x_{2,\alpha(2)} \cdots x_{n,\alpha(n)},$$

$$\mathrm{per}(X) = \sum_{\alpha \in S_n} x_{1,\alpha(1)} x_{2,\alpha(2)} \cdots x_{n,\alpha(n)}.$$

Theorem 4 (Zhang, Jin, Yan, 2023)

Let $X = (x_{st})_{n \times n}$ be a matrix of order n over the complex field and let m be the number of non-zero entries of X. Then the permanent $\mathrm{per}(X)$ of X satisfies:

$$(m - n)\mathrm{per}(X) = \sum_{(i,j) \in I} \mathrm{per}(X_{ij}), \tag{4}$$

where $I = \{(i,j) | x_{ij} \neq 0, 1 \leq i,j \leq n\}$.

- For convenience, we assume that $\{x_{11}, x_{12}, \ldots, x_{nn}\}$ is a set of n^2 independent commuting variables over the complex field.
- There exist $n!$ terms in the expansion of the φ-immanant $\mathrm{Imm}_\varphi(X)$ of X, denoted by $\mathcal{T}_{\alpha_1}, \mathcal{T}_{\alpha_2}, \ldots, \mathcal{T}_{\alpha_{n!}}$, where $S_n = \{\alpha_i | 1 \leq i \leq n\}$ is the symmetric group of order n and

$$\mathcal{T}_{\alpha_i} = \varphi(\alpha_i) x_{1\alpha_i(1)} x_{2\alpha_i(2)} \cdots x_{n\alpha_i(n)}.$$

- Hence

$$\mathrm{Imm}_\varphi(X) = \sum_{i=1}^{n!} \mathcal{T}_{\alpha_i}.$$

- Define an $n! \times n^2$ matrix $T = (t_{\alpha_i x_{st}})_{n! \times n^2}$ as

$$t_{\alpha_i x_{st}} = \begin{cases} \mathcal{T}_{\alpha_i}, & \text{if } (s,t) \notin \{(k, \alpha_i(k)) | 1 \leq k \leq n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Theorem 1

- Hence each row of T has $n^2 - n$ entries each of which equals $\mathcal{T}_{\alpha_i}$ and n entries each of which equals zero.

- For example, if $n = 3$, then $U = S_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and $V = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}\}$, where

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \alpha_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Proof of Theorem 1

$$\mathcal{T}_{\alpha_1} = \varphi(\alpha_1)x_{11}x_{22}x_{33}, \mathcal{T}_{\alpha_2} = \varphi(\alpha_2)x_{11}x_{23}x_{32},$$
$$\mathcal{T}_{\alpha_3} = \varphi(\alpha_3)x_{13}x_{22}x_{31}, \mathcal{T}_{\alpha_4} = \varphi(\alpha_4)x_{12}x_{21}x_{33},$$
$$\mathcal{T}_{\alpha_5} = \varphi(\alpha_5)x_{12}x_{23}x_{31}, \mathcal{T}_{\alpha_6} = \varphi(\alpha_6)x_{13}x_{21}x_{32}.$$

$$T = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & x_{31} & x_{32} & x_{33} \end{array} \\ \begin{pmatrix} 0 & \mathcal{T}_{\alpha_1} & \mathcal{T}_{\alpha_1} & \mathcal{T}_{\alpha_1} & 0 & \mathcal{T}_{\alpha_1} & \mathcal{T}_{\alpha_1} & \mathcal{T}_{\alpha_1} & 0 \\ 0 & \mathcal{T}_{\alpha_2} & \mathcal{T}_{\alpha_2} & \mathcal{T}_{\alpha_2} & \mathcal{T}_{\alpha_2} & 0 & \mathcal{T}_{\alpha_2} & 0 & \mathcal{T}_{\alpha_2} \\ \mathcal{T}_{\alpha_3} & \mathcal{T}_{\alpha_3} & 0 & \mathcal{T}_{\alpha_3} & 0 & \mathcal{T}_{\alpha_3} & 0 & \mathcal{T}_{\alpha_3} & \mathcal{T}_{\alpha_3} \\ \mathcal{T}_{\alpha_4} & 0 & \mathcal{T}_{\alpha_4} & 0 & \mathcal{T}_{\alpha_4} & \mathcal{T}_{\alpha_4} & \mathcal{T}_{\alpha_4} & \mathcal{T}_{\alpha_4} & 0 \\ \mathcal{T}_{\alpha_5} & 0 & \mathcal{T}_{\alpha_5} & \mathcal{T}_{\alpha_5} & \mathcal{T}_{\alpha_5} & 0 & 0 & \mathcal{T}_{\alpha_5} & \mathcal{T}_{\alpha_5} \\ \mathcal{T}_{\alpha_6} & \mathcal{T}_{\alpha_6} & 0 & 0 & \mathcal{T}_{\alpha_6} & \mathcal{T}_{\alpha_6} & \mathcal{T}_{\alpha_6} & 0 & \mathcal{T}_{\alpha_6} \end{pmatrix} \end{array}.$$

- Note that the i-th row in matrix T has exactly $n^2 - n$ non-zero entries each of which equals $\mathcal{T}_{\alpha_i}$. Hence the sum of entries of i-th row in T equals $(n^2 - n)\mathcal{T}_{\alpha_i}$. So the sum of all entries in T equals $(n^2 - n)\mathrm{Imm}_{\varphi}(X)$. That is,

$$\sum_{i=1}^{n!} \sum_{1 \leq s,t \leq n} t_{\alpha_i x_{st}} = (n^2 - n)\mathrm{Imm}_{\varphi}(X). \tag{5}$$

- On the other hand, by the definition of X_{st}, the sum of entries of x_{st}-th column of T equals exactly the determinant of X_{st}. Hence

$$\sum_{1 \leq s,t \leq n} \sum_{i=1}^{n!} t_{\alpha_i x_{st}} = \sum_{1 \leq s,t \leq n} \mathrm{Imm}_{\varphi}(X_{st}). \tag{6}$$

- Hence, by Eqs. (5) and (6), the theorem holds.

Outline

- Let $G = (V(G), E(G))$ be a digraph having no loops and no multiple arcs, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G) = \{e_1, e_2, \ldots, e_m\}$.

- Denote the adjacency matrix and the vertex in-degree diagonal matrix of G by $A = (a_{ij})_{n\times n}$ and $D = diag(d^+(v_1), d^+(v_2), \cdots, d^+(v_n))$, where $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$ otherwise, and $d^+(v)$ is the number of arcs with head v_i.

- Then $D-A$ and $D+A$ are the Laplacian and signless Laplacian matrices of G.

Six Digraph Polynomials

- Set

$$
\begin{aligned}
f_1(G;x) = & \ \det(xI - A), \\
f_2(G;x) = & \ \det(xI - D + A), \\
f_3(G;x) = & \ \det(xI - D - A), \\
f_4(G;x) = & \ \mathrm{per}(xI - A), \\
f_5(G;x) = & \ \mathrm{per}(xI - D + A), \\
f_6(G;x) = & \ \mathrm{per}(xI - D - A),
\end{aligned}
$$

  where $\det(X)$ and $\mathrm{per}(X)$ denote the determinant and permanent of a square matrix X.

- Then $f_1(G;x)$, $f_2(G;x)$, $f_3(G;x)$, $f_4(G;x)$, $f_5(G;x)$, and $f_6(G;x)$ are the characteristic polynomial, Laplacian characteristic polynomial, signless Laplacian characteristic polynomial, permanental polynomial, Laplacian permanental polynomial, signless Laplacian permanental polynomial of digraph G, respectively.

Outline

- Suppose that both β and γ are real numbers satisfying $\gamma \neq 0$. Let G be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G) = \{e_1, e_2, \ldots, e_m\}$, which has no loops or multiple arcs. Set

$$g_1(G; x) = \det(xI_n - \beta D - \gamma A),$$

$$g_2(G; x) = \mathrm{per}(xI_n - \beta D - \gamma A).$$

Theorem 5 (Zhang, Jin, Yan, 2023)

Both $g_1(G; x)$ and $g_2(G; x)$ defined as above satisfy:

$$(m - n)g_1(G; x) + xg_1'(G; x) = \sum_{e \in E(G)} g_1(G - e; x), \tag{7}$$

$$(m - n)g_2(G; x) + xg_2'(G; x) = \sum_{e \in E(G)} g_2(G - e; x). \tag{8}$$

Theorem 6 (Zhang, Jin, Yan, 2023)

Let G be a digraph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and arc set $E(G) = \{e_1, e_2, \dots, e_m\}$, which has no loops or multiple arcs. Then, for any $i = 1, 2, 3, 4, 5, 6,$

$$(m-n)f_i(G;x) + xf_i'(G;x) = \sum_{e \in E(G)} f_i(G-e;x). \tag{9}$$

Theorem 7 (Zhang, Jin, Yan, 2023)

Let G be a digraph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and arc set $E(G) = \{e_1, e_2, \dots, e_m\}$, which has no loops or multiple arcs. If $m \neq n$, then $f_i(G;x)$ can be reconstructed from $\{f_i(G-e;x) | e \in E(G)\}$ for $1 \leq i \leq 6$.

$\{f_i(G-e;x) | e \in E(G)\} \Rightarrow f_i(G;x)$ for $1 \leq i \leq 6$ if $m \neq n$.

- Let $G_1 = C_n$ be a directed cycle with vertex set $V(G_1) = \{v_i | 1 \leq i \leq n\}$ and arc set $E(G_1) = \{(v_i, v_{i+1}) | 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$.
- Let G_2 be the digraph obtained from G_1 by replacing arc (v_n, v_1) with (v_1, v_n).

$$\{f_i(G_1 - e; x) | e \in E(G_1)\} = \{f_i(G_2 - e) | e \in E(G_2)\},$$

$$\text{but } f_i(G_1; x) \neq f_i(G_2; x) \text{ for } i = 1, 4.$$

Theorem 8 (Zhang, Jin, Yan, 2023)

If $m = n$, the characteristic polynomial $f_1(G; x)$ and permanental polynomial $f_4(G; x)$ of a digraph G with n vertices and n arcs can not be determined uniquely by $\{f_1(G - e; x) | e \in E(G_1)\}$ and $\{f_4(G - e; x) | e \in E(G_1)\}$, respectively. That is, $\{f_i(G - e; x) | e \in E(G)\} \nRightarrow f_i(G; x)$ for $i = 1, 4,$ if $m = n$.

- Note that, $f_2(G;x) = \det(xI - D + A)$. Hence $f_2(G;0) = \det(A - D) = 0$. If $m = n$, then

$$xf_2'(G;x) = \sum_{e \in E(G)} f_2(G - e; x).$$

Given the initial condition $f_2(G;0) = 0$, the differential equation above has a unique solution.

Theorem 9 (Zhang, Jin, Yan, 2023)

The Laplacian characteristic polynomial $f_2(G;x) = \det(xI - D + A)$ *of a digraph* G *with arc set* $E(G)$ *can be reconstructed from* $\{f_2(G - e; x) | e \in E(G)\}$. *That is,* $\{f_2(G - e; x) | e \in E\} \Rightarrow f_2(G;x)$.

Outline

Discussion

- For a digraph $G = (V(G), E(G))$,

$$\{f_i(G - e; x) | e \in E(G)\} \Rightarrow f_i(G; x) \text{ for } 1 \le i \le 6, \text{ if } m \ne n.$$

$$\{f_i(G - e; x) | e \in E(G)\} \nRightarrow f_i(G; x) \text{ for } i = 1, 4, \text{ if } m = n.$$

$$\{f_2(G - e; x) | e \in E(G)\} \Rightarrow f_2(G; x), \text{ if } m = n.$$

$$\{f_i(G - e; x) | e \in E(G)\} \Rightarrow f_i(G; x) \text{ for } i = 3, 5, 6, \text{ if } m = n?$$

- **A natural problem is: Do the above results hold for simple graphs?**
- Let G be a simple graph with adjacency matrix A and vertex degree diagonal matrix D. Set

$$\begin{aligned} h_1(G;x) &= \det(xI - A), \\ h_2(G;x) &= \det(xI - D + A), \\ h_3(G;x) &= \det(xI - D - A), \\ h_4(G;x) &= \mathrm{per}(xI - A), \\ h_5(G;x) &= \mathrm{per}(xI - D + A), \\ h_6(G;x) &= \mathrm{per}(xI - D - A). \end{aligned}$$

Theorem 10 (Zhang, Jin, Yan, Liu, 2023)

$$(m-n)h_1(G;x)+xh_1'(G;x) = \sum_{uv\in E(G)} \left[h_1(G-uv;x)+h_1(G-u-v;x)\right], \tag{10}$$

$$(m-n)h_4(G;x)+xh_4'(G;x) = \sum_{uv\in E(G)} \left[h_4(G-uv;x)-h_4(G-u-v;x)\right], \tag{11}$$

and for $j=2,3$,

$$(m-n)h_j(G;x)+xh_j'(G;x) = \sum_{uv\in E(G)} h_j(G-uv;x). \tag{12}$$

Discussion

- For a graph $G = (V, E)$, if $m \neq n$, then

$$\{h_i(G-e;x)|e \in E\} \cup \{h_i(G-u-v;x)|uv \in E\} \Rightarrow h_i(G;x) \text{ if } i = 1, 4,$$

$$\{h_i(G-e;x)|e \in E\} \Rightarrow h_i(G;x) \text{ for } i = 2, 3.$$

Thank you for your attention!