Phase Transitions of Structured Codes of Graphs

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Overview

- What are Structured Codes of Graphs?
- Known Results
- An Interesting Phenomenon
- Our Results

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- Instead of prescribing the minimum distance between codewords, the authors require the codewords differ in some specific structure.

What are Structured Codes of Graphs?

- Def : Let $[n]$ denote $\{1, 2, \ldots, n\}$. A graph G on $[n]$ can be viewed as a codeword (i.e. $\{0, 1\}$-sequence) of length $\binom{n}{2}$ by using edges to represent 1 and non-edges to represent 0. Then a family of graphs on $[n]$ can be viewed as a $\{0, 1\}$-code.

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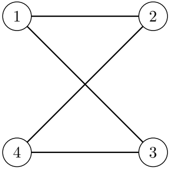

Figure: An Example

This 4-vertex graph is a $\{0, 1\}$-sequence of length 6, 110011.

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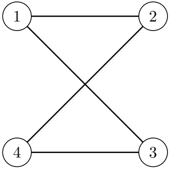

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- Def : The symmetric difference of two graphs G and H on $[n]$, denoted by $G \oplus H$, is the graph on $[n]$ whose edge set is just the symmetric difference of $E(G)$ and $E(H)$.

What are Structured Codes of Graphs?

- Def : Let $\mathcal{F}$ be a fixed class of graphs. A family $\mathcal{G}$ of graphs on $[n]$ is called $\mathcal{F}$-**good**, if the symmetric difference of any two members in $\mathcal{G}$ belongs to $\mathcal{F}$. This family $\mathcal{G}$ is also called an $\mathcal{F}$-code.

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- Def : Let $M_{\mathcal{F}}(n)$ denote the maximum possible size of an $\mathcal{F}$-good family on $[n]$.

- Def : If the graph class $\mathcal{F}$ consists of all graphs containing a fixed graph L, then we use $M_L(n)$ and L-code instead of $M_{\mathcal{F}}(n)$ and $\mathcal{F}$-code. In another word, $M_L(n)$ denotes the maximum possible size of a family $\mathcal{G}$ of graphs on $[n]$ such that the symmetric difference of any two members in $\mathcal{G}$ contains L as a subgraph.

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- A Simple Proof : Let $G_1, G_2, \ldots, G_s$ forms an $\mathcal{F}$-good family and $H_1, H_2, \ldots, H_t$ forms a graph family such that the symmetric difference of any two members in it doesn't belongs to $\mathcal{F}$.

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- Then $G_i \oplus H_j$ are pairwise different. Because $(G_i \oplus H_j) \oplus (G_p \oplus H_q) = (G_i \oplus G_p) \oplus (H_j \oplus H_q)$.

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Theorem (Alon et. al.(2023))

$M_{\mathcal{F}_s}(n) = n + 1$ for all odd n and $M_{\mathcal{F}_s}(n) = n$ for all even n.

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Problem 1

For what "natural" sequences $\{T_i\}_{i\geq 1}$ of trees (with T_i having exactly i vertices for every i) will the value of $M_{T_n}(n)$ grow only linearly in n? A similar question is valid if T_i is replaced by $\mathcal{T}_i$, some "natural" family of i-vertex trees.

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Theorem (Theorem I)

For infinitely many n and all integers $3 \leq \ell \leq \frac{n-1}{12\log n} + 2$, we have $M_{\mathcal{F}_\ell}(n) \geq 2^{n-2}$. In particular, this holds whenever $n \geq 64$ and $n = p$ or $n = 2p - 1$ for odd primes p.

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- Remark : This theorem shows that the family $\mathcal{T}_\ell$ consisting of all spanning trees with ℓ leaves for any $2 \le \ell \le \frac{n-1}{12\log n} + 2$ can not provide a positive answer to Problem 1.

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Perfect 1-factorization Conjecture, Kotzig(1964)

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- This conjecture is still open in general, but it is known to hold in several special cases. For example, whenever $n = p + 1$ (Kotzig(1964)) or $n = 2p$ for some odd prime p (Anderson(1973) and Nakamura(1975)).

Proof of Theorem I

- The proof consists of two parts :
  - Construct a graph family $\mathcal{G}$ of size 2^{n-2} such that the symmetric difference of any two members in $\mathcal{G}$ contains a Hamiltonian path and at least $3\ell - 5$ disjoint additional edges.
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- For the first part, let $n \geq 65$ be an odd integer such that the perfect 1-factorization conjecture holds for $n+1$, then we can partition the edge set of K_{n+1} into n perfect matchings $M_1, M_2, \ldots, M_n$ such that the union of any two of them forms a Hamiltonian cycle in K_{n+1}.

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- For each $1 \leq i \leq n$, we delete the edge adjacent to $n+1$ in M_i, then for any $i \neq j \in [n]$, $M_i \cup M_j$ forms a Hamiltonian path in K_n.

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- Let $\mathcal{G}_i = \{G \cup H_i | G \in \mathcal{G}'_i\}$ for all $1 \leq i \leq s$ and $\mathcal{G} = \bigcup_{i=1}^{s} \mathcal{G}_i$. Then we get our desired $\mathcal{G}$.

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- For the second part. Let $T = v_1 v_2 \dots v_n$ be the Hamiltonian path consists of 2 matchings in $\mathcal{M}$ and let E_A be the set of $3\ell - 5$ additional edges. We first remove the edge adjacent to v_n from E_A if there exists such an edge in E_A and then do the following operation:

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- Note that after this operation, the number of leaves in the spanning tree T increases exactly one, and we remove at most 3 edges from E_A.
- So we can repeat this operation $\ell - 2$ times and then T becomes a spanning tree with exactly ℓ leaves.

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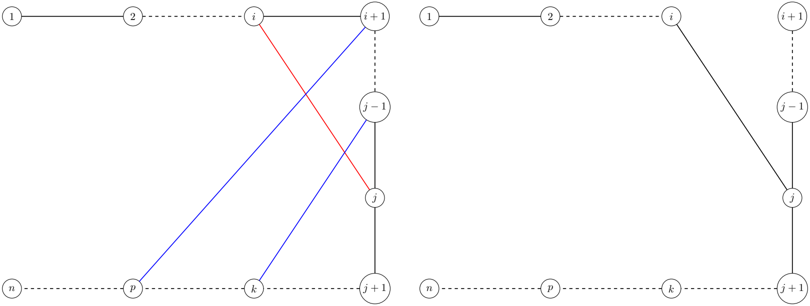

(a) Before the operation (b) After the operation

Our Results

- Def : For a given graph L, let $M_L(n, k)$ denote the largest cardinality of a family $\mathcal{G}$ of graphs on $[n]$, such that the symmetric difference of any two members of $\mathcal{G}$ contains at least k copies of L. Let $v(L)$ and $e(L)$ denote the number of vertices and edges in L, respectively.

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Theorem (Theorem II)

Let L be any graph with at least one edge. If $k = o(n^{v(L)})$, then we have

$$\lim_{n\to\infty} \frac{\log M_L(n,k)}{\binom{n}{2}} = \frac{1}{\chi(L)-1}.$$

If $k = cn^{v(L)}$ for some constant $c > 0$, then we have

$$\lim_{n\to\infty} \frac{\log M_L(n,k)}{\binom{n}{2}} \leq \frac{1}{\chi(L)-1} - \frac{2c}{e(L)}.$$

Proof of Theorem II

- Def : Fix a graph L. A graph G is called *L-free* if G does not contain L as a subgraph. Let the Turán number of L, denoted by $ex(n, L)$, be the maximum number of edges in an n-vertex L-free graph.

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Theorem (Erdös-Stone(1946))

For any graph L with at least one edge, we have

$$ex(n, L) = \left(1 - \frac{1}{\chi(L) - 1} + o(1)\right)\binom{n}{2}.$$

Proof of Theorem II

Theorem (Erdös-Frankl-Rödl(1986))

For any fixed graph L, if $\chi(L) = r \geq 3$, then
$F_n(L) = 2^{ex(n,K_r)(1+o(1))} = 2^{\binom{n}{2}(1-\frac{1}{\chi(L)-1}+o(1))}$.

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Theorem (Graph Removal Lemma(1986))

For any fixed graph L and any $\varepsilon > 0$, there exists $\delta > 0$, such that if an n-vertex graph G contains at most $\delta n^{v(L)}$ copies of L, then we can remove at most εn^2 edges of G to get an L-free graph.

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- Let $F_n(L, k)$ denote the number of graphs on $[n]$ containing at most $k-1$ copies of L. By the Graph Removal Lemma, we have $F_n(L, k) \leq F_n(L)\binom{\binom{n}{2}}{o(n^2)} = 2^{\binom{n}{2}(1-\frac{1}{\chi(L)-1}+o(1))}$.

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- $\alpha(G_L) \geq |V(G_L)|/(\Delta(G_L)+1) = 2^{\binom{n}{2}(\frac{1}{\chi(L)-1}+o(1))}$.

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- For the case $k = cn^{v(L)}$, we construct a graph which contains at most $k-1$ copies of L by adding edges to an extremal L-free graph.

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- Thus, we can obtain a lower bound of the corresponding dual concept $D_L(n, k)$ by constructing a family consisting of all subgraphs of H.
- Recall that $M_L(n,k)D_L(n,k) \leq 2^{\binom{n}{2}}$. Therefore,

$$M_L(n,k) \leq 2^{\left(\frac{1}{\chi(L)-1} - \frac{2c}{e(L)} + o(1)\right)\binom{n}{2}}$$

for $k = cn^{v(L)}$.

Our Results

- Def : For any fixed graph L, let $M_{k \cdot L}(n)$ denote the largest size of a graph family $\mathcal{G}$ on $[n]$, such that the symmetric difference of any two members of $\mathcal{G}$ contains at least k vertex-disjoint copies of L.

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Theorem (Theorem III)

Let L be any graph with at least one edge. If $k = o(n)$, then we have

$$\lim_{n\to\infty} \frac{\log M_{k \cdot L}(n)}{\binom{n}{2}} = \frac{1}{\chi(L) - 1}.$$

If $k = cn$ for some constant $c > 0$, then we have

$$\lim_{n\to\infty} \frac{\log M_{k \cdot L}(n)}{\binom{n}{2}} \le \frac{(1-c)^2}{\chi(L) - 1}.$$

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- Since in an arbitrary graph, a fixed copy of L intersects at most $v(L)n^{v(L)-1}$ different copies of L. A graph, which contains at most $k-1$ vertex-disjoint copies of L, contains at most $(k-1)(v(L)n^{v(L)-1}+1) = o(n^{v(L)})$ different copies of L.

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- Because we already know that the number of graphs which contains at most $o(n^{v(L)})$ copies of L is $2^{(1-\frac{1}{\chi(L)-1}+o(1))\binom{n}{2}}$, the number of graphs which contains at most $k-1$ vertex-disjoint copies of L is also $2^{(1-\frac{1}{\chi(L)-1}+o(1))\binom{n}{2}}$.

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- By the same argument in the previous proof, we have

$$\lim_{n\to\infty} \frac{\log M_{k\cdot L}(n)}{\binom{n}{2}} = \frac{1}{\chi(L)-1}$$

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- Then G contains at most $k - 1$ vertex-disjoint copies of L and $e(G) = (1 - \frac{(1-c)^2}{\chi(L)-1})\binom{n}{2} + o(n^2)$. Thus we can obtain a lower bound of the corresponding dual concept $D_{k \cdot L}(n)$ by constructing a family consisting of all subgraphs of G.
- Therefore, $M_{k \cdot L}(n) \leq 2^{(\frac{(1-c)^2}{\chi(L)-1} + o(1))\binom{n}{2}}$ for $k = cn$.

Our Results

Theorem (Theorem IV)

If $t = o(\log n)$, *then we have*

$$\lim_{n\to\infty} \frac{\log M_{K_{t,t}}(n)}{\binom{n}{2}} = 1.$$

If $t = c\log n$ *for some constant* $c > 0$, *then we have*

$$\lim_{n\to\infty} \frac{\log M_{K_{t,t}}(n)}{\binom{n}{2}} \leq 1 - 2^{-\frac{2}{c}}.$$

Proof of Theorem IV

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For any integer $t \geq s \geq 2$, we have

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- So we have $2^{ex(n,K_{t,t})} \leq F_n(K_{t,t}) \leq \binom{\binom{n}{2}}{o(n^2)}$. Thus, $F_n(K_{t,t}) = 2^{o(1)\binom{n}{2}}$.

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- By the same argument in the proof of Theorem II and Theorem III, we have $M_{K_{t,t}}(n) = 2^{(\frac{1}{\chi(L)-1}+o(1))\binom{n}{2}}$ for $k = o(\log n)$.

Proof of Theorem IV

- It remains to consider the case when $t = c \log n$ for some constant $c > 0$. And in this case, we only need to construct an n-vertex $K_{t,t}$-free graph G with at least $2^{-\frac{2}{c}}\binom{n}{2}$ edges.

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- Let $\delta = 2^{-\frac{2}{c}}$ and consider the Erdös-Rényi random graph $G(n, \delta)$ (i.e. an n-vertex graph in which each possible edge is present independently with probability δ).
- Let X be the number of $K_{t,t}$ in $G(n, \delta)$, we have

$$\mathbb{E}[X] = \frac{1}{2}\binom{n}{2t}\binom{2t}{t}\delta^{t^2} < n^{2t}\delta^{t^2} = (n^2\delta^t)^t.$$

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- Since $\delta^t = 2^{-\frac{2}{c}c\log n} = n^{-2}$, we have $\mathbb{E}[X] < 1$.

Proof of Theorem IV

- By average, there exists a graph G' such that $e(G') - X \geq \mathbb{E}[e(G(n,\delta)) - X] > \delta\binom{n}{2} - 1$.

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- Let G be obtained from G' by deleting one edge for each copy of $K_{t,t}$ in G', then G is an n-vertex $K_{t,t}$-free graph with at least $2^{-\frac{2}{c}}\binom{n}{2}$ edges.

Our Results

Theorem (Theorem V)

Let $L(r, m) = (A \cup B, E)$ be a connected bipartite graph on m vertices such that any vertex in A has at most r neighbors in B. If $m = O(n^{1-\varepsilon})$ for some constant $\varepsilon > 0$, then for any constant integer r, we have

$$\lim_{n\to\infty} \frac{\log M_{L(r,m)}(n)}{\binom{n}{2}} = 1.$$

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Let $\alpha \in (0,1)$, t, r, m, u, n be integers such that $\alpha^t n - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u$. Then for any n-vertex graph G with at least $\frac{\alpha}{2}n^2$ edges, there exists $U \subseteq V(G)$ with $|U| \geq u$ such that any r-set $S \subseteq U$ has at least m common neighbors in G.

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- Firstly, we claim that $ex(n, L(r,m)) = o(n^2)$.
- Let $\alpha = n^{-\frac{\varepsilon^2}{2r}}$, $t = \frac{r}{\varepsilon}$ and $u = m$. Then for sufficiently large n, we have $\alpha^t n - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u$.

Proof of Theorem V

- So, for sufficiently large n and any n-vertex graph G with at least $\frac{1}{2}n^{2-\frac{\varepsilon^2}{r}}$ edges, there exists $U \subseteq V(G)$ with $|U| \geq u$ such that any r-set $S \subseteq U$ has at least m common neighbors.

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- Let ϕ be any injection from B to U, we only need to extend it to an injection from $A \cup B$ to $V(G)$ such that for any edge ab in $L(r,m)$, $\phi(a)\phi(b)$ is an edge in G.

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- Let A' be a subset of A and assume that we have already extend ϕ to an injection from $A' \cup B$ to $V(G)$. Take an vertex $v \in A \setminus A'$, then $\phi(N_{L(r,m)}(v))$ is a subset of U with cardinality at most r.

Proof of Theorem V

- Take an r-set $S \subseteq U$ with $S' \subseteq S$ and let T denote the set of common neighbors of S in G. Then $|T| \geq m = |V(L(r, m))|$.

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- Since for sufficiently large n, any n-vertex graph G with at least $\frac{1}{2}n^{2-\frac{\varepsilon^2}{r}}$ edges contains a copy of $L(r,m)$, we know that $ex(n, L(r,m)) = O(n^{2-\frac{\varepsilon^2}{r}}) = o(n^2)$. Then by the same argument in Theorem II, we have $\lim_{n\to\infty} \frac{\log M_{L(r,m)}(n)}{\binom{n}{2}} = 1$.

Our Results

- Since any graph on vertex set $[n]$ can be viewed as a spanning subgraph of K_n, what if we replace K_n with some alternative graphs on $[n]$?

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- The most natural instance of this problem that comes to mind is when $\mathcal{F}$ denotes the family of all connected graphs and G denotes an $m \times n$ grid.
- Def : An $m \times n$ grid, denoted by $G_{m,n}$, is the graph with vertex set $[m] \times [n]$ and with edges between (u, v) and (i, j) if and only if $u = i$ and $v \equiv j \pm 1 (\mathrm{mod}\ n)$ or $v = j$ and $u \equiv i \pm 1 (\mathrm{mod}\ m)$.

Our Results

Proposition VI

For any integers $m, n \geq 3$, let $M_{\mathcal{F}_c}(G_{m,n})$ denote the maximum possible size of a family $\mathcal{G}$ of spanning subgraphs of $G_{m,n}$ such that the symmetric difference of any two members in $\mathcal{G}$ is connected, then we have $M_{\mathcal{F}_c}(G_{m,n}) \leq 16$. Especially, we also have $M_{\mathcal{F}_c}(G_{m,n}) = 16$ for $m, n \leq 4$.

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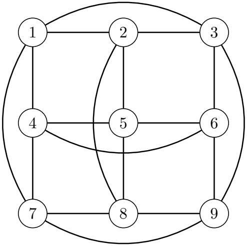
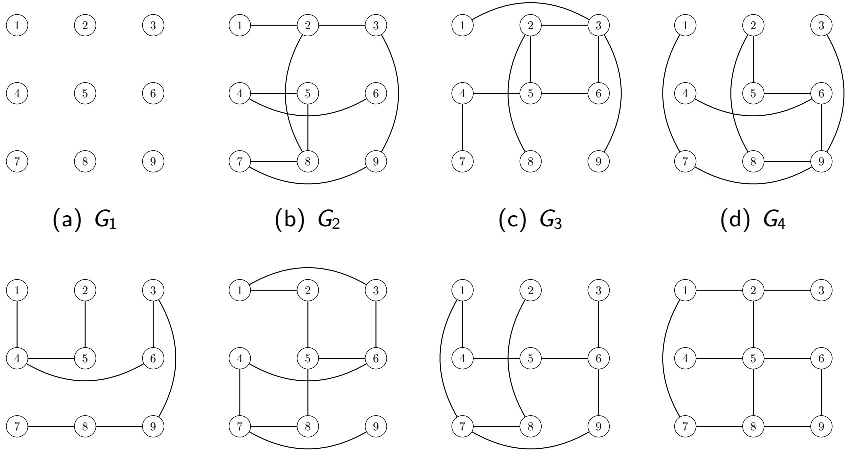


Figure: 3×3 grid

- Remark : It is natural to ask whether the upper bound is also sharp for all $m, n \geq 4$. We are seeking a general construction of it.

Construction for $m = n = 3$



(a) G_1 (b) G_2 (c) G_3 (d) G_4

(e) G_5 (f) G_6 (g) G_7 (h) G_8

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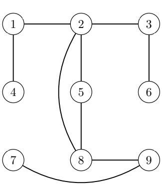
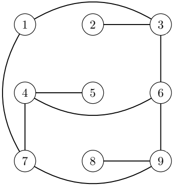
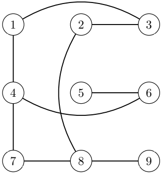


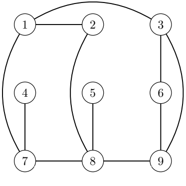
(i) G_9



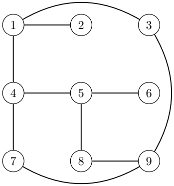
(j) G_{10}



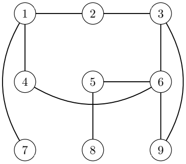
(k) G_{11}



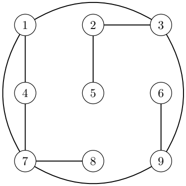
(l) G_{12}



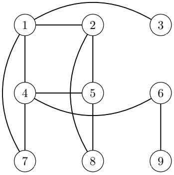
(m) G_{13}



(n) G_{14}



(o) G_{15}



(p) G_{16}

Thanks!

Thank you for your attention!