Anti-ramsey number of matchings in 3-graphs

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Outline

Notations and Terminologies

- Let $[n] := \{1, 2, \cdots, n\}$ and $\binom{[n]}{k} := \{T \subseteq [n] : |T| = k\}$.
- A n-vertex k-uniform hypergraph H is a pair $H = (V, E)$, where $V := [n]$ and $E \subseteq \binom{[n]}{k}$. A k-uniform hypergraph is also called a k-graph for short.
- A matching is a set of pairwise disjoint edges. The size of the largest matching in H is denoted by $\nu(H)$. A matching is perfect if it covers all vertices of H.

- An edge-coloring of a hypergraph H is a mapping $\phi : E(H) \to \{1, 2, \ldots, c\}$
- An edge-colored hypergraph H is called rainbow if every edge of H receives a different color.
- The anti-Ramsey number $ar(n, k, H)$ is the smallest integer c such that each edge-coloring of the n-vertex k-uniform complete hypergraph with exactly c colors contains a rainbow copy of H.

- The Turán number $ex(n, k, H)$ is the maximum possible number of edges in an n-vertex k-graph which does not contain H as a subgraph.
- For a set of graphs $\mathcal{H}$, the Turán number $ex(n, k, \mathcal{H})$ is the maximum possible number of edges in an n-vertex k-graph which does not contain any $H \in \mathcal{H}$ as a subgraph.

Proposition 1.

$$2 + ex(n, k, \mathcal{G}) \leq ar(n, k, G) \leq 1 + ex(n, k, G),$$

where $\mathcal{G} = \{G - e : e \in E(G)\}$.

Proposition 2.

$$2 + ex(n, k, M_{s-1}) \leq ar(n, k, M_s) \leq 1 + ex(n, k, M_s).$$

Theorem [Erdős, Simonovits, Sós, 1973]

For sufficiently large n, $ar(n, 2, k_p) = ex(n, 2, k_{p-1}) + 2$.

Theorem [Montellano-Ballesteros and Neumann-Lara, 2002]

For all $n \geq p \geq 4$, $ar(n, 2, k_p) = ex(n, 2, k_{p-1}) + 2$.

P. Erdős, M. Simonovits, V.T. Sós, Anti-Ramsey theorems, in: Infinite and Finite sets, Vol. II, in: Colloq. Math. Soc. János Bolyai, vol. 10, 1975, pp. 633-643. Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday.

J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem, *Combinatorica*, **22** (2002), 445–449.

- Schiermeyer proved that $ar(n, 2, M_s) = ex(n, 2, M_{s-1}) + 2$ for $s \geq 2$ and $n \geq 3s + 3$.
- Fujita, Kaneko, Schiermeyer and Suzuki established the same result for $s \geq 2$ and $n \geq 2s + 1$.
- Chen, Li and Tu determined the exact value of $ar(n, 2, M_s)$ for all $s \geq 2$ and $n \geq 2s$.

I. Schiermeyer, Rainbow numbers for matchings and complete graphs, *Discrete Math.*, **286** (2004), 157–162.

S. Fujita, A. Kaneko, I. Schiermeyer and K. Suzuki, A rainbow k-matching in the complete graph with r colors, *Electron. J. Combin.*, **16** (2009), R51.

H. Chen, X. Li and J. Tu, Complete solution for the rainbow numbers of matchings, *Discrete Math.*, **309** (2009), 3370-3380.

S. Fujita, C. Magnant, and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.*, **26**(1) (2010), 1-30.

Gu, Li, Shi determined the anti-Ramsey number of Loose paths/cycles and linear paths/cycles in k-graph for sufficiently large n, and bounds of anti-Ramsey number for Berge paths/cycles.

R. Gu, J. Li, and Y. Shi, Anti-Ramsey numbers of paths and cycles in hypergraphs, *SIAM J. Discrete Math.*, **34**(1) (2020), 271-307.

- Let $\mathbb{P}_k$ denote the linear path of length k, and $\mathbb{C}_k$ denote the linear cycle of length k.
- Let $\mathcal{P}_k$ denote the family of loose paths of length k, and $\mathcal{C}_k$ denote the family of loose cycles of length k.

Theorem (Gu, Li, Shi 2020)

For any integer k, if $k = 2t \geq 4$ and $s \geq 3$, then for sufficiently large n,

$$ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t + 1 > 5$ and $s \geq 4$, then for sufficiently large n,

$$ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2.$$

Theorem (Gu, Li, Shi 2020)

$ar(n, s, \mathbb{C}_k) = ar(n, s, \mathbb{P}_k)$ for $k = 2t \geq 8$, $s \geq 4$ and sufficiently large n; $ar(n, s, \mathbb{C}_k) = ar(n, s, \mathbb{P}_k)$ for $k = 2t + 1 \geq 11$, $s \geq k + 3$ and sufficiently large n;

Theorem (Gu, Li, Shi 2020)

For any integer k, if $k = 2t \geq 4$ and $s \geq 3$, then for sufficiently large n,

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if $k = 2t + 1 \geq 5$ and $s \geq 3$, then for sufficiently large n,

$$ar(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 3.$$

Theorem (Gu, Li, Shi 2020)

$ar(n, s, \mathcal{C}_k) = ar(n, s, \mathcal{P}_k)$ for $k = 2t \geq 8$, $s \geq 4$ and sufficiently large n; $ar(n, s, \mathcal{C}_k) = ar(n, s, \mathcal{P}_k)$ for $k = 2t + 1 \geq 11$, $s \geq k + 3$ and sufficiently large n;

Tang, Li, Yan determined $ar(n, r, \mathbb{P}_k)$ and $ar(n, r, \mathbb{C}_k)$ for all $k \geq 3$ and $r \geq 3$ except $ar(n, s, \mathbb{C}_3)$.

Y. Tang, T. Li, and G. Yan, Anti-Ramsey numbers of expansions of paths and cycles in uniform hypergraphs, *J. Graph Theory,* **101**(4) (2022), 668-685.

Definition

- Given a 2-graph G, the expansion of G is an r-graph on $|V(G)| + (r-2)|E(G)|$ vertices obtained from G by adding $r-2$ vertices to each edge in G.
- Denote the expansion of $K_{\ell+1}$ by $H^r_{\ell+1}$.

- Liu and Song determined the anti-Ramsey number of $H^r_{\ell+1}$.
- Liu and Song determined the anti-Ramsey number of F^3. F^3 is the expansion of F, where F is obtained from a tree T of a certain class of trees by adding a new edge.

X. Liu and J. Song, Hypergraph anti-Ramsey theorems, arXiv:2310.01186.

X. Liu and J. Song, Exact results for some extremal problems on expansions I, arXiv:2310.01736.

Conjecture (Özkahya and Young 2013)

Let $k \geq 3$, $s \geq 3$. If $n > ks$, then

$$ar(n, k, M_s) = ex(n, k, M_{s-1}) + 2.$$

In addition, if $n = ks$, then

$$ar(n, k, M_s) = \begin{cases} ex(n, k, M_{s-1}) + 2, & \text{if } s < c_k \\ ex(n, k, M_{s-1}) + k + 1, & \text{if } s \geq c_k \end{cases}$$

where c_k is a constant dependent on k.

L. Özkahya and M. Young, Anti-Ramsey number of matchings in hypergraphs, *Discrete Math.*, **313** (2013), 2359–2364.

- Jin determined the exact value of the anti-Ramsey number of a k-matching in a 3-partite 3-uniform complete hypergraph for $n_3 \geq n_2 \geq n_1 \geq 3k-2$.
- Xue, Shan, Kang proved a multi-partite version of Özkahya and Young's conjecture.

Z. Jin, Anti-Ramsey number of matchings in a hypergraph, *Discrete Math.*, **344** (2021), 112594.

Y. Xue, E. Shan and L. Kang, Anti-Ramsey number of matchings in r-partite r-uniform hypergraphs, *Discrete Math.*, **345** (2022), 112782.

Frankl and Kupavskii proved Özkahya and Young's conjecture for all $n \geq sk + (s-1)(k-1)$ and $k \geq 3$.

Theorem (Frankl and Kupavskii 2019)

$ar(n, k, M_s) = ex(n, k, M_{s-1}) + 2$ for $n \geq sk + (s-1)(k-1)$ and $k \geq 3$.

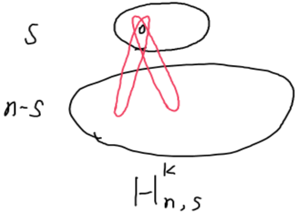
P. Frankl and A. Kupavskii, Two problems on matchings in set families - in the footsteps of Erdős and Kleitman, *J. Combin. Theory ser. B*, **138** (2019), 286–313.

Erdős Matching Conjecture and Stability

Erdős matching conjecture (1965)

For $n \geq k(s+1) - 1$ we have

$$ex(n, k, M_{s+1}) = \max\left\{|H^k_{n,s}|, |D^k_{n,s}|\right\}.$$



Erdős matching conjecture (1965)

For $n \geq k(s+1) - 1$ we have

$$ex(n, k, M_{s+1}) = \max\left\{|H^k_{n,s}|, |D^k_{n,s}|\right\}.$$

- $n \geq 2k^3 s$
- $n \geq 3k^2 s$

B. Bollobás, D. E. Daykin, and P. Erdős, Sets of independent edges of a hypergraph, *Quart. J. Math. Oxford Ser.*, **27** (1976), 25–32.

H. Huang, P. Loh and B. Sudakov, The size of a hypergraph and its matching number, *Combin. Probab. Comput.*, **21** (2012), 442–450.

- $n \geq (2s+1)k - s$
- $n \geq \frac{5}{3}sk - \frac{2}{3}s.$

P. Frankl, Improved bounds for Erdős Matching Conjecture, *J. Combin. Theory Ser. A*, **120** (2013), 1068–1072.

P. Frankl and A. Kupavskii, The Erdős matching conjecture and concentration inequalities, *J. Combin. Theory Ser. B*, **157** (2022), 366–400.

- $k(s+1) \leq n \leq (k+\varepsilon)(s+1)$

P. Frankl, Proof of the Erdős matching conjecture in a new range, *Israel J. Math.*, **222** (2017), 421-430.

- $k = 3$

T. Łuczak, and K. Mieczkowska, On Erdős extremal problem on matchings in hypergraphs, *J. Combin. Theory Ser. A*, **124** (2014), 178–194.

P. Frankl, On maximum number of edges in a hypergraph with given matching number, *Discrete Appl. Math.*, **216** (2017), 562-581.

Theorem [Frankl and Kupavskii 2019]

Suppose that $k \geq 3$ and either $n \geq (s + \max\{25, 2s+2\})k$ or $n \geq (2 + o(1))sk$, where $o(1)$ is with respect to $s \to \infty$. Then for any k-graph H with $\nu(H) \leq s$, if

$$|H| > \binom{n}{k} - \binom{n-s}{k} - \binom{n-s-k}{k-1} + 1,$$

then H is a subgraph of $H^k_{n,s}$.

Definition

Given two k-graphs H_1, H_2 and a real number $\varepsilon > 0$, we say that H_2 is *ε-contains* to H_1 if $V(H_1) = V(H_2)$ and $|E(H_1) \backslash E(H_2)| \leq \varepsilon |V(H_1)|^k$.

Stability Lemma [Guo, Lu and Mao 2022]

Let ε, ρ be two reals such that $0 < \rho \ll \varepsilon < 1$. Let n, s be two integers such that n is sufficiently large and $n/54 + 1 \leq s \leq 13n/45 + 1$. Let H be a 3-graph on vertex set $[n]$. If $e(H) > ex(n, 3, M_s) - \rho n^3$ and $\nu(H) \leq s - 1$, then H ε-contains $H^3_{n,s-1}$ or $D^3_{n,s-1}$.

M. Guo, H. Lu, and D. Mao, A stability result on matchings in 3-uniform hypergraphs, *SIAM J. Discrete Math.*, **36** (2022), 2339-2351.

Our Results

Theorem 1.

For sufficiently large n, the following holds

$$ar(n, 3, M_s) = \begin{cases} ex(n, 3, M_{s-1}) + 2, & \text{if } 3s < n < 5s - 2; \\ ex(n, 3, M_{s-1}) + 5, & \text{if } n = 3s. \end{cases}$$

Lower bound

$$ar(n,3,M_s) \geq \begin{cases} ex(n,3,M_{s-1}) + 2, & \text{if } 3s < n < 5s-2; \\ ex(n,3,M_{s-1}) + 5, & \text{if } n = 3s. \end{cases}$$

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and let $f : E(K_n^3[U]) \to [\binom{|U|}{3}]$ be a bijective coloring.

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Let

$$\phi(e) = \begin{cases} f(e), & e \in E(K_n^3[U]); \\ 0, & \text{otherwise.} \end{cases}$$

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$$\phi(e) = \begin{cases} f(e), & e \in E(K_n^3[U]); \\ 0, & \text{otherwise.} \end{cases}$$

$$|\phi(e)| = |D_{n,s-1}^3| + 1 \geq ex(n, 3, M_{s-1}) + 1$$

Lower bound

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$$\phi(e) = \begin{cases} f'(e), & e \in P; \\ 0, & \text{otherwise.} \end{cases}$$

$$|\phi(e)| = |H_{n,s-1}^3| + 1 \geq ex(n, 3, M_{s-1}) + 1$$

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Let $U \subseteq V(K_n^3)$ be a set of size $(3s-4)$ and let $f : E(K_n^3[U]) \rightarrow [\binom{|U|}{3}]$ be a bijective coloring.

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Let $W = \{1,2,3,4\} = V(K_n^3) \setminus U$. Let $A_1 = \{\{1,2\},\{3,4\}\}$ $A_2 = \{\{1,3\},\{2,4\}\}$, $A_3 = \{\{1,4\},\{2,3\}\}$.

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Let

$$\phi(e) = \begin{cases} f(e), & e \in E(K_n^k[U]); \\ \binom{|U|}{3} + i, & e \cap W \subseteq A_i; \\ 0, & \text{otherwise.} \end{cases}$$

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- $n = 3s$

Let $U \subseteq V(K_n^3)$ be a set of size $(3s-4)$ and let $f : E(K_n^3[U]) \to [\binom{|U|}{3}]$ be a bijective coloring.

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Let

$$\phi(e) = \begin{cases} f(e), & e \in E(K_n^k[U]); \\ \binom{|U|}{3} + i, & e \cap W \subseteq A_i; \\ 0, & \text{otherwise.} \end{cases}$$

$ar(n,3,M_s) \geq |\phi(e)| + 1 = ex(n,3,M_{s-1}) + 5.$

Upper bound

Lemma 2.

For a given real $0 < c_0 \ll 1$, there exists an integer $n_0 = n_0(c_0)$ such that $ar(n, 3, M_s) \leq ex(n, 3, M_{s-1}) + 2$ for $n/6 \leq s \leq (1 - c_0)n/3$ and $n > n_0$.

Sketch of proof of Lemma 2

$$ex(n, 3, M_{s-1}) = \max\{\binom{n}{3} - \binom{n-s+2}{3}, \binom{3s-4}{3}\}.$$

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Let $c(n,s) := \max\{\binom{n}{3} - \binom{n-s+2}{3}, \binom{3s-4}{3}\} + 2$.

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Let $c(n,s) := \max\{\binom{n}{3} - \binom{n-s+2}{3}, \binom{3s-4}{3}\} + 2$.

Let $f_{n,s} : E(K_n^3) \to [c(n,s)]$ be a surjective coloring.

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Let $f_{n,s} : E(K_n^3) \to [c(n, s)]$ be a surjective coloring.

Let G be a subgraph of H with $c(n, s)$ edges such that each color appears on exactly one edge of G.

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Let G be a subgraph of H with $c(n, s)$ edges such that each color appears on exactly one edge of G.

By stability lemma, G ε-contains $H^3_{n,s-1}$ or $D^3_{n,s-1}$ for $n/6 \leq s \leq 13n/45 + 1$.

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By stability lemma, G ε-contains $H^3_{n,s-1}$ or $D^3_{n,s-1}$ for $n/6 \le s \le 13n/45 + 1$.

Case 1. $n/6 \le s \le 13n/45 + 1$ and G ε-contains $H^3_{n,s-1}$.

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By stability lemma, G ε-contains $H^3_{n,s-1}$ or $D^3_{n,s-1}$ for $n/6 \leq s \leq 13n/45 + 1$.

Case 1. $n/6 \leq s \leq 13n/45 + 1$ and G ε-contains $H^3_{n,s-1}$.

Case 2. $n/6 \leq s \leq 5n/18$ and G ε-contains $D^3_{n,s-1}$.

Sketch of proof of Lemma 2

By stability lemma, G ε-contains $H^3_{n,s-1}$ or $D^3_{n,s-1}$ for $n/6 \leq s \leq 13n/45 + 1$.

Case 1. $n/6 \leq s \leq 13n/45 + 1$ and G ε-contains $H^3_{n,s-1}$.

Case 2. $n/6 \leq s \leq 5n/18$ and G ε-contains $D^3_{n,s-1}$.

Case 3. $5n/18 + 1 \leq s \leq 13n/45 + 1$ and G ε-contains $D^3_{n,s-1}$ or $13n/45 + 2 \leq s \leq (1 - c_0)n/3$.

Sketch of proof of Lemma 2

Key Lemma

Given reals $0 < \varepsilon \ll c_0 \ll 1$, there exists an integer n_0 such that the following holds. Let H be a 3-graph with $n > n_0$ vertices. Let s be an integer. If $\nu(H) \leq s$ and

$$e(H) > \binom{3s+1}{3} + 3s(n - 3s - 1), \tag{3.1}$$

then the following holds.

1. For $5n/18 - 1 \leq s \leq 13n/45$, if H ε-contains $D^3_{n,s}$, then H is a subgraph of $D^3_{n,s}$.
2. For $13n/45 \leq s \leq (1 - c_0)n/3$, H is a subgraph of $D^3_{n,s}$.

Lemma 3.

For a given real $0 < c_0 \ll 1$, there exists an integer $n_0 = n_0(c_0)$ such that for $n > n_0$,

$$ar(n, 3, M_s) \leq \begin{cases} \binom{3s-4}{3} + 2, & \text{if } (1 - c_0)n/3 \leq s < n/3 \ ; \\ \binom{3s-4}{3} + 5, & \text{if } s = n/3. \end{cases}$$

Sketch of proof of Lemma 3

Let $f_{n,s} : E(K_n^3) \to [c(n,s)]$ be a surjective coloring.

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Let G be a subgraph of H with $c(n,s)$ edges such that each color appears on exactly one edge of G.

Sketch of proof of Lemma 3

Let $V(H) = [n]$ such that $d_G(1) \geq d_G(2) \geq \cdots \geq d_G(n)$.

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Let $V(H) = [n]$ such that $d_G(1) \geq d_G(2) \geq \cdots \geq d_G(n)$.

- Step 1: Let $U := [3s-4]$ and
  $R := \{x \in U : d_{G[U]}(x) < n^2/15\}$, then $r < 2c_0 n$.

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Let $V(H) = [n]$ such that $d_G(1) \geq d_G(2) \geq \cdots \geq d_G(n)$.

- Step 1: Let $U := [3s-4]$ and $R := \{x \in U : d_{G[U]}(x) < n^2/15\}$, then $r < 2c_0n$.

Let $H' := H - E(H[U \setminus R])$.

Sketch of proof of Lemma 3

- Step 2: If H' has a rainbow matching M such that $|V(M) \cap (W \cup R)| \geq r + 4$, then G has a rainbow matching of size s, where $W = [n] \setminus U$.

Sketch of proof of Lemma 3

- Step 3: H' has a rainbow matching M such that $|V(M) \cap (W \cup R)| \geq r + 4$

The Future Work

- The case $n = ks$.
- The anti-Ramsey number of expansion of some graph.

Let U be a subset of $V(K_n^k)$ such that $|U| = n - k - 1$ and let $W := V(K_n^k) \setminus U$. Thus $|W| = k + 1$. Let $f : E(K_n^k[U]) \to [\binom{|U|}{k}]$ be a bijective coloring.

Let U be a subset of $V(K_n^k)$ such that $|U| = n - k - 1$ and let $W := V(K_n^k) \setminus U$. Thus $|W| = k + 1$. Let $f : E(K_n^k[U]) \to [\binom{|U|}{k}]$ be a bijective coloring.

For an odd integer k, there are $\frac{1}{2}\binom{k+1}{\frac{k+1}{2}}$ distinct subsets $A_1, \ldots, A_{\frac{1}{2}\binom{k+1}{\frac{k+1}{2}}}$ of W such that $|A_i| = (k+1)/2$ for $1 \le i \le \frac{1}{2}\binom{k+1}{\frac{k+1}{2}}$ and $A_i \cap A_j \neq \emptyset$ for $1 \le i < j \le \frac{1}{2}\binom{k+1}{\frac{k+1}{2}}$. Let $\mathcal{A}_i := \{e \in E(K_n^k) : e \cap W = A_i \text{ or } e \cap W = W \setminus A_i\}$ and let $\mathcal{H}_1$ be the complete k-graph K_n^k with edge coloring $f_{\mathcal{H}_1}$, where

$$f_{\mathcal{H}_1}(e) = \begin{cases} f(e), & e \in E(K_n^k[U]); \\ \binom{|U|}{k} + i, & e \in \mathcal{A}_i \text{ for } 1 \le i \le \frac{1}{2}\binom{k+1}{\frac{k+1}{2}}; \\ 0, & \text{otherwise.} \end{cases}$$

For an even integer k, let $x \in W$. There are $\binom{k}{\frac{k}{2}-1}$ distinct subsets $B_1, \ldots, B_{\binom{k}{\frac{k}{2}-1}}$ of $W \setminus \{x\}$ such that $|B_i| = k/2 - 1$ for $1 \leq i \leq \binom{k}{\frac{k}{2}-1}$. Let $\mathcal{B}_i := \{e \in E(K_n^k) : x \in e \text{ and } e \cap W = B_i\} \cup \{e \in E(K_n^k) : e \cap W = W \setminus (B_i \cup \{x\})\}$ and let $\mathcal{H}_2$ be the n-vertex complete k-graph K_n^k with edge coloring $f_{\mathcal{H}_2}$, where

$$f_{\mathcal{H}_2}(e) = \begin{cases} f(e), & e \in E(\mathcal{H}_2[U]); \\ \binom{|U|}{k} + i, & e \in \mathcal{B}_i \text{ for } 1 \leq i \leq \binom{k}{\frac{k}{2}-1}; \\ 0, & \text{otherwise.} \end{cases}$$

$$ar(n,k,s) \geq \begin{cases} \binom{n-k-1}{k} + \frac{1}{2}\binom{k+1}{\frac{k+1}{2}} + 2, & k \text{ is odd;} \\ \binom{n-k-1}{k} + \binom{k}{\frac{k}{2}-1} + 2, & k \text{ is even.} \end{cases}$$

Thank you!