Some Problems in Extremal Set Theory

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Outline

- Cover-free hypergraphs
- Union-free hypergraphs
- Cancellative hypergraphs

Notations

- $[n] = \{1, \ldots, n\}$
- $\binom{[n]}{r} = \{A \subseteq [n] : |A| = r\}$
- r-uniform hypergraph (or r-graph) $\mathcal{H} \subseteq \binom{[n]}{r}$

Definitions

$\mathcal{H} \subseteq \binom{[n]}{r}$ is called

- (Kautz-Singleton'64; Erdős-Frankl-Füredi'85) t-cover-free if for any $t+1$ distinct edges $A_1, \ldots, A_t, B \in \mathcal{H}$,

$$B \not\subseteq \cup_{i=1}^{t} A_i;$$

- (Kautz-Singleton'64; Erdős-Moser'70) t-union-free if for any two distinct subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{H}$, with $1 \leq |\mathcal{A}|, |\mathcal{B}| \leq t$,

$$\cup_{A \in \mathcal{A}} A \neq \cup_{B \in \mathcal{B}} B;$$

- (Erdős-Katona'74; Füredi'12) t-cancellative if for any $t+2$ distinct edges $A_1, \ldots, A_t, B, C \in \mathcal{H}$,

$$(\cup_{i=1}^{t} A_i) \cup B \neq (\cup_{i=1}^{t} A_i) \cup C.$$

Applications beyond combinatorics

- testing blood samples, chemical leak, electric shorting, Denial-of-Service attack, etc
- multi-access channel communication
- DNA screening
- pattern finding
- digital and multimedia fingerprinting
- $\cdots$
- In information/coding theory, cover-free hypergraph is equivalent to
    - non-adaptive group testing
    - disjunct matrix
    - superimposed code

Goal of this talk

- Let $F_t(n, r)$, $U_t(n, r)$, and $C_t(n, r)$ denote the maximum size of t-cover-free, t-union-free, and t-cancellative r-graphs on n vertices, respectively.

- Study the asymptotic behaviour of these functions for fixed r, t, and $n \to \infty$.[1]

- Focus on the Turán exponents, is there[2] $\alpha \in [0, r]$ s.t. $g_t(n, r) = \Theta(n^\alpha)$, where $g \in \{F, U, C\}$?

[1] One can also study the problem for fixed t and $r, n \to \infty$, very different

[2] Usually α doesn't exist or $\alpha < r$, the problem is degenerate.

Cover-free r-graphs

- $\mathcal{H} \subseteq \binom{[n]}{r}$ is t-cover-free if for any $t+1$ distinct edges $A_1, \ldots, A_t, B \in \mathcal{H}$,
$$B \not\subseteq \cup_{i=1}^{t} A_i.$$
- "Trivial" example: $r = 1$, singletons
- "Non-trivial" example: (n, r, k)-packings, an n-vertex r-graph whose every two edges have $< k$ intersections
- $(n, r, \lceil \frac{r}{t} \rceil)$-packings are automatically t-cover-free
- (Rödl'85) $\exists$ (n, r, k)-packings with size $\geq (1 - o(1)) \left(\binom{n}{k} / \binom{r}{k}\right)$

Cover-free r-graphs - upper and lower bounds

- $(n, r, \lceil \frac{r}{t} \rceil)$-packings: $F_t(n,r) \geq (1-o(1)) \left(\binom{n}{\lceil \frac{r}{t} \rceil} / \binom{r}{\lceil \frac{r}{t} \rceil} \right)$
- (Erdős-Frankl-Füredi'85) $\lim_{n\to\infty} \frac{F_t(n,r)}{n^{\lceil \frac{r}{t} \rceil}}$ exists (and determined it for some parameters).
- (Frankl-Füredi'87) $\lim_{n\to\infty} \frac{F_t(n,r)}{\binom{n}{\lceil \frac{r}{t} \rceil}} = \frac{1}{\binom{r}{\lceil \frac{r}{t} \rceil} - m(r, \lceil \frac{r}{t} \rceil, \ell)}$, where
    1. $0 \leq \ell < t$ is the unique integer s.t. $r = t(\lceil \frac{r}{t} \rceil - 1) + \ell + 1$,
    2. $m(r, k, \ell)$ is Erdős hypergraph matching function, which is the maximum number of edges of an r-vertex k-graph containing no $\ell + 1$ pairwise disjoint edges.

Conjecture of Erdős-Frankl-Füredi'85

- Consider not necessarily uniform t-cover-free families $\mathcal{H} \subseteq 2^{[n]}$
- Let $N(t) = \min\{n : \exists \ t\text{-}CFF \ with \ \geq n+1 \ edges\}$
- "Trivial" example: singletons form a t-cover-free families with n edges
- Question. What is $N(t)$?

CFFs rising from affine planes

Definition (Affine plane)

Affine plane of order $t+1$, where $t+1$ is a prime power, is a point-line incidence structure $(\mathcal{P}, \mathcal{L})$ s.t.

- $|\mathcal{P}| = (t+1)^2$, $|\mathcal{L}| = (t+1)^2 + (t+1)$
- every line contains $t+1$ points
- every point is contained in $t+2$ lines
- every pair of lines intersect in at most one point

- Affine plane of order $t+1$ gives a t-CFF $(t+1)$-graph with $(t+1)^2$-vertices and $(t+1)^2 + (t+1)$ edges
- For prime power $t+1$, $N(t) \leq (t+1)^2$; for all $t \to \infty$, $N(t) \leq t^2 + o(t^2)$.

Conjecture of Erdős-Frankl-Füredi'85, continued

- Conjecture. (Erdős-Frankl-Füredi'85):
  $\lim_{t\to\infty} \frac{N(t)}{t^2} = 1$ (weak version) and $N(t) \geq (t+1)^2$ (strong version)
- (Erdős-Frankl-Füredi'85) $N(t) \geq \frac{5}{6}t^2 + o(t^2)$
- (S.-Ge'16) $N(t) \geq \frac{15+\sqrt{33}}{24}t^2 > 0.86n^2$, based on Erdős-Gallai'59 exact formula $m(r,2,\ell) = \max\{\binom{2\ell+1}{2}, \binom{r}{2} - \binom{r-\ell}{2}\}$ for $r \geq 2\ell+2$.

Union-free r-graphs

- $\mathcal{H} \subseteq \binom{[n]}{r}$ is t-union-free if for any two distinct subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{H}$, with $1 \leq |\mathcal{A}|, |\mathcal{B}| \leq t$, $\cup_{A \in \mathcal{A}} A \neq \cup_{B \in \mathcal{B}} B$
- Observation. t-cover-free $\Longrightarrow$ t-union-free $\Longrightarrow$ $(t-1)$-cover-free
- Bounds via CFF. $\Omega(n^{\lceil \frac{r}{t} \rceil}) = F_t(n, r) \leq U_t(n, r) \leq F_{t-1}(n, r) = O(n^{\lceil \frac{r}{t-1} \rceil})$.
- Improved bounds
  1. (Frankl-Füredi'86) $U_2(n, r) = \Theta(n^{\lceil 4r/3 \rceil / 2})$
  2. (Füredi-Ruszinkó'13) $n^{2-o(1)} < U_r(n, r) = O(n^2)$
     - Conjecture: $U_r(n, r) = o(n^2)$, no Turán exponent
  3. (S.-Tamo'20) $\Omega(n^{\frac{r}{t-1}}) = U_t(n, r)$, exact Turán exponent for $(t-1) \mid r$.

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  4. (S.-Tamo'20) $\gcd(r, t-1) = 1$, $\Omega\left(n^{\frac{r}{t-1}} (\log n)^{\frac{1}{t-1}}\right) = U_t(n,r)$

Cancellative r-graphs

- $\mathcal{H} \subseteq \binom{[n]}{r}$ is t-cancellative if for any $t+2$ distinct edges $A_1, \ldots, A_t, B, C \in \mathcal{H}$, $(\cup_{i=1}^{t} A_i) \cup B \neq (\cup_{i=1}^{t} A_i) \cup C$
- Observation. $(t+1)$-cover-free $\Longrightarrow$ t-cancellative $\Longrightarrow$ "almost" $\lfloor \frac{t}{2} \rfloor$-cover-free
- Bounds via CFF. $\Omega(n^{\lceil \frac{r}{t+1} \rceil}) = C_t(n,r) = O(n^{\lceil \frac{r}{\lfloor \frac{t}{2} \rfloor} \rceil})$.
- Improved bounds
  1. (Tolhuizen'00, Frankl-Füredi'84) $\frac{0.28}{2^r}\binom{n}{r} < C_1(n,r) \leq \frac{2^r}{\binom{2r}{r}}\binom{n}{r}$
  2. (Füredi'12) $\Omega(n^{\lfloor \frac{r}{2} \rfloor}) = C_2(n,r) = O(n^{\lceil \frac{r}{2} \rceil})$
  3. (S.-Tamo'20) $\Omega(n^{\lfloor \frac{2r}{t+2} \rfloor + \frac{2r \pmod{t+2}}{t+1}}) = C_t(n,r) = O(n^{\lceil \frac{r}{\lfloor t/2 \rfloor + 1} \rceil})$, exact Turán exponent for $2 \mid t$ and $(t/2+1) \mid r$.

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  4. (S.-Tamo'20) $\gcd(2r - \lceil \frac{2r-t-1}{t+2} \rceil, t+1) = 1$, $\Omega\left(n^{\lfloor \frac{2r}{t+2} \rfloor + \frac{2r \pmod{t+2}}{t+1}} (\log n)^{\frac{1}{t+1}}\right) = C_t(n,r)$

Sparse r-graphs

- (v, e)-free r-graphs: the union of any e distinct edges contains at least $v + 1$ vertices, i.e, for $\forall$ distinct edges $A_1, \ldots, A_e \in \mathcal{H}$,

$$|\cup_{i=1}^{e} A_i| \geq v + 1$$

- Example: Fano plane is $(4, 2)$-free, $(5, 3)$-free, but not $(6, 4)$-free

Sparse r-graphs - existence

Lemma (Brown-Erdős-Sós'73, standard probabilistic construction)

Let $s \geq 1, r \geq 3$ and (v_i, e_i) for $1 \leq i \leq s$ be pairs of integers satisfying $v_i \geq r+1, e_i \geq 2$. Let $h := \min\left\{\frac{e_i r - v_i}{e_i - 1} : 1 \leq i \leq s\right\}$. Then there exists an r-graph with $\Omega(n^h)$ edges that is simultaneously (v_i, e_i)-free for each $1 \leq i \leq s$.

Lemma (S.-Tamo'20, cancellative r-graphs via sparse r-graphs)

$\forall\ 0 \le x \le r-1$,
$(tr+x, t+2)$-*free* + $(2r-x-1, 2)$-*free* $\Longrightarrow$ t-*cancellative.*

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- $(2r-x-1, 2)$-free $\Longrightarrow |B \cap C| \le x$

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- So $(B \cup C) \backslash (B \cap C) \subseteq \cup_{i=1}^{t} A_i$
- $(2r-x-1, 2)$-free $\Longrightarrow |B \cap C| \le x$
- Therefore

$$\begin{aligned} |(\cup_{i=1}^{t} A_i) \cup B \cup C| &= |\cup_{i=1}^{t} A_i| + |(B \cup C) \setminus (\cup_{i=1}^{t} A_i)| \\ &\le tr + |B \cap C| \\ &\le tr + x, \end{aligned}$$

a contradiction to $(tr+x, t+2)$-free.

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r-partite + $(tr-r,t)$-free + $(tr,2t)$-free $\Longrightarrow$ t-union-free

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Fact

If an r-partite r-graph with vertex parts $V_1, \ldots, V_r$ is $(tr-r, t)$-free, then $\forall$ distinct edges $A_1, \ldots, A_t$, $\exists$ i s.t. $A_1 \cap V_i, \ldots, A_t \cap V_i$ all distinct.

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- $(tr-r, t)$-free $\Longrightarrow |\mathcal{A}| = |\mathcal{B}| = t$

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- $(tr,2t)$-free $\Longrightarrow$ $\mathcal{A}\cap\mathcal{B}\neq\emptyset$, $|\mathcal{A}\cap\mathcal{B}|=i\ge 1$
- $\mathcal{A}:=\{C_1,\dots,C_i,A_{i+1},\dots,A_t\}$ and $\mathcal{B}:=\{C_1,\dots,C_i,B_{i+1},\dots,B_t\}$,

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- $\mathcal{A}:=\{C_1,\dots,C_i,A_{i+1},\dots,A_t\}$ and $\mathcal{B}:=\{C_1,\dots,C_i,B_{i+1},\dots,B_t\}$,
- Apply Fact to $\mathcal{A}':=\mathcal{A}\cup\{B_{i+1}\}\backslash\{C_1\}$, $\exists$ $1\le j\le r$ such that $A\cap V_j, A\in\mathcal{A}'$ are pairwise distinct.

Summary

- Cover-free, union-free, cancellative hypergraphs
- Old bounds:
  - t-cover-free $\Longrightarrow$ t-union-free $\Longrightarrow$ $(t-1)$-cover-free
  - $(t+1)$-cover-free $\Longrightarrow$ t-cancellative $\Longrightarrow$ "almost" $\lfloor \frac{t}{2} \rfloor$-cover-free
- New bounds:
  - r-partite $+$ $(tr-r, t)$-free $+$ $(tr, 2t)$-free $\Longrightarrow$ t-union-free
  - $(tr+x, t+2)$-free $+$ $(2r-x-1, 2)$-free $\Longrightarrow$ t-cancellative
- Questions.
  1. Minimum n, s.t. $\exists$ t-CFF $\mathcal{H} \subseteq 2^{[n]}$ with $|\mathcal{H}| > n$?
  2. $\Omega(n^{\max\left\{\frac{r}{t-1}, \lceil \frac{r}{t} \rceil\right\}}) = U_t(n,r) = O(n^{\lceil \frac{r}{t-1} \rceil})$, correct exponents?
  3. $\Omega(n^{\lfloor \frac{2r}{t+2} \rfloor + \frac{2r \pmod{t+2}}{t+1}}) = C_t(n,r) = O(n^{\lceil \frac{r}{\lfloor t/2 \rfloor + 1} \rceil})$, correct exponents?
  4. Maximum cardinality with no size restriction on the edges?

$\mathscr{THANK}$

$\mathscr{YOU}$