Basics on the hypergraph container method

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May 7, 2020

Introduction

This is a gentle introduction to basics of the hypergraph container method introduced independently by Balogh, Samotij and Morris, and Saxton and Thomason about 10 years ago.

The method has seen numerous applications in extremal combinatorics and other related areas in the past decade. We will focus mostly on examples, illustrating how to apply this method on various types of problems.

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Key idea

Independent sets in many 'natural' hypergraphs are 'clustered' together.

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Warm up of estimating the number of triangle-free graphs on $[n]$.

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Theorem [Erdős-Kleitman-Rothschild (1976)]

The number is triangle-free graphs on $[n]$ is

$$2^{n^2/4+o(n^2)} = 2^{\mathrm{ex}(n,K_3)+o(n^2)}.$$

First attempt, count triangle-free graphs

- Lower bound: take all subgraphs of $K_{n/2,n/2}$, at least $2^{n^2/4}$.

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- Upper bound: $\mathrm{ex}(n, K_3) = n^2/4$, so at most

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Problem: Inefficient to count them one by one; lot of them are similar.

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- (with supersaturation) $\forall G \in \mathcal{F}$, $e(G) \leq n^2/4 + o(n^2)$.

Supersaturation for triangles

For any $\epsilon > 0$, there is $\delta > 0$ such that

$$e(G) \geq \left(\frac{1}{4} + \epsilon\right) n^2 \implies \delta n^3 \text{ triangles}$$

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The number of triangle-free graphs on n is at most

$$\sum_{G \in \mathcal{F}} 2^{e(G)} \leq |\mathcal{F}| \cdot 2^{\max_{G \in \mathcal{F}}\{e(G)\}} \leq 2^{n^2/4+o(n^2)}.$$

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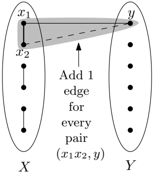

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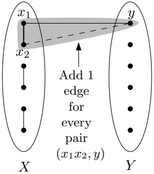

$n/4$ matching edges in X, $n/2$ vertices in $Y \Rightarrow 2^{n/4 \cdot n/2} = 2^{n^2/8}$.

Matching upper bound

Theorem[Balogh-Petrickova 2014]

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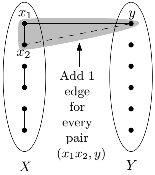
Theorem[Balogh-L.-Petrickova-Sharifzadeh 2015]

Almost all maximal triangle-free graphs looks like the ones from the construction.



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Triangle removal lemma

If an n-vertex G has $o(n^3)$ triangles, then it can be made triangle-free by removing $o(n^2)$ edges.

Upper bound

Fix $G \in \mathcal{F}$, Removal lemma implies $E(G) = E(G_1) \cup E(G_2)$, with triangle-free G_1 and $e(G_2) = o(n^2)$.

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#(S1) is negligible: $2^{o(n^2)}$. Suffices to show, given H_2, the number of its extensions is at most $2^{n^2/8}$.

Bound (S2) via auxiliary graph

Triangle-free H_2 and G_1, define link graph $L = L_{H_2}(G_1)$:

$$V(L) := E(G_1)$$
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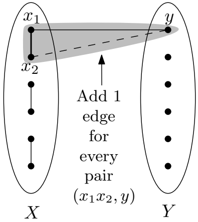

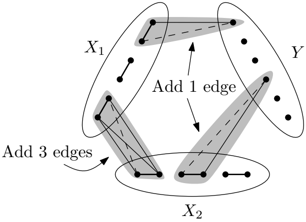

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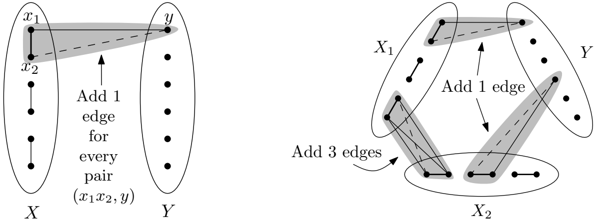

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- Upper bound: Improvement: $\forall r \geq 3\ \exists \epsilon_r > 0$: $\leq 2^{\mathrm{ex}(n,K_{r+1})-\epsilon_r n^2}$.

Cameron-Erdős Conjecture

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A set $S \subseteq [n]$ is sum-free if $x + y \notin S$ for every $x, y \in S$ (x and y are not necessarily distinct), i.e. no solution to $x + y = z$.

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Green (2004), Sapozhenko (2003)

There are constants c_0 and c_1, s.t. the number of sum-free subsets of $[n]$ is

$$(1+o(1))c_i 2^{n/2},$$

where $i = n \mod 2$.

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- Suppose that $4|n$ and set $I_1 := \{n/2+1, \ldots, 3n/4\}$ and $I_2 := \{3n/4+1, \ldots, n\}$. First choose the element $n/4$ and a set $S \subseteq I_2$. Then for every $x \in I_2 \setminus S$, choose $x - n/4 \in I_1$. No further element in I_2 can be added.

Upper bound

Denote by $f_{\max}(n)$ the number of maximal sum-free subsets in $[n]$.
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Łuczak-Schoen (2001)

$$f_{\max}(n) = O(2^{n/2-2^{-28}n}).$$

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$$f_{\max}(n) = O(2^{n/2-2^{-28}n}).$$

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$$f_{\max}(n) \leq 2^{3n/8+o(n)}.$$

Upper bound

Denote by $f_{\max}(n)$ the number of maximal sum-free subsets in $[n]$. Recall that $f_{\max}(n) \geq 2^{n/4}$.

Cameron-Erdős Conjecture

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Balogh-L.-Sharifzadeh-Treglown (2018)

For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \mod 4$, $[n]$ contains $(C_i + o(1))2^{n/4}$ maximal sum-free sets.

Tools

From additive number theory:

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- Container Lemma of Green and Ruzsa.

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- Not too sparse and almost regular graphs.

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Container Lemma for sum-free sets [Green-Ruzsa]

There exists $\mathcal{F} \subseteq 2^{[n]}$, s.t.

- (container) $\forall S \subseteq [n]$ sum-free, $\exists F \in \mathcal{F}$, s.t. $S \subseteq F$;
- (few) $|\mathcal{F}| = 2^{o(n)}$;
- (almost sum-free) $\forall F \in \mathcal{F}$ contains $o(n^2)$ Schur triples.
- (with supersaturation) $\forall F \in \mathcal{F}$, $|F| \leq (1/2 + o(1))n$.

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- (with supersaturation) $\forall F \in \mathcal{F}$, $|F| \leq (1/2 + o(1))n$.

Suffices to show that for every container $A \in \mathcal{F}$,

$$f_{\max}(A) \leq 2^{n/4+o(n)}.$$

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For any $A \subseteq [n]$ with $o(n^2)$ Schur triples, $A = B \cup C$ where B is sum-free and $|C| = o(n)$.

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Structural Lemma

$A \subseteq [n]$ with $o(n^2)$ Schur triples and $|A| = (\frac{1}{2} - \gamma)n$ with $\gamma = \gamma(n) \leq 1/11$, then
(i) (interval-like) $A = B \cup C$ where $|C| = o(n)$ and $B \subseteq [(1/2 - \gamma)n, n]$.
(ii) (odds-like) Almost all elements of A are odd, i.e. $|A \setminus O| = o(n)$.

Sketch of the proof

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Given $S, B \subseteq [n]$ sum-free, the link graph of S on B is $L_S[B]$, where $V = B$ and $x \sim y$ iff $\exists z \in S$ s.t. $\{x, y, z\}$ is a Schur triple.

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Lemma

Given $S, B \subseteq [n]$ sum-free and $I \subseteq B$, if $S \cup I$ is a maximal sum-free subset of $[n]$, then I is a maximal independent set in $L_S[B]$.

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- For a fixed S, $\#$ extensions in (2) is exactly $\mathrm{MIS}(L_S[B])$,

$$\mathrm{MIS}(L_S[B]) \leq 3^{|B|/3} \leq 3^{0.45n/3} \ll 2^{0.249n}.$$

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$s = b_1 - b_2$, $s' = b_1 - b_3$ and $s'' = b_2 - b_3$ are in S,
but then $s + s'' = s'$.

Hujter-Tuza: G triangle-free $\Rightarrow \mathrm{MIS}(G) \leq 2^{|G|/2}$.

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- Large $|S| \geq n^{1/4} \Rightarrow L_S[B]$ is almost regular and dense.
  - $\forall G : \delta(G) \to \infty$, $\Delta(G) \leq k\delta(G)$ $\Rightarrow \mathrm{MIS}(G) \leq 3^{\left(\frac{k}{k+1}\right)\frac{|G|}{3}+o(|G|)}$.

Thank you!