Container May 14

Today's plan

1. Recall last time
   one simple example
2. See 3-unif version hypergraph container
3. See graph containers
   - Alg. (Kleitman - Winston) 80s

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1) **Container Thm (Δ-free graphs)**

$\exists$ family $\mathcal{C}$ of graphs s.t.

(i) $\forall F \in \mathcal{F}_n(K_3)$, $\exists C \in \mathcal{C}$ s.t. $F \subseteq C$

(ii) $|\mathcal{C}| \leq n^{O(n^{3/2})}$ → size of fingerprint

(iii) $\forall C \in \mathcal{C}$, almost Δ-free

$\mathcal{F}_n(K_3)$ = {all Δ-free graphs on $[n]$}

(iv) (+ Supersaturation)

$$\forall\, C \in \mathcal{C}, \quad e(C) \leq \frac{n^2}{4} + o(n^2)$$

<u>Phrase this in terms of indep. in</u> <u>hypergraphs</u> (more abstract / general)

Consider $\mathcal{H}$ 3-unif $\begin{cases} V(\mathcal{H}) = E(K_n) \\ e_1 e_2 e_3 \in E(\mathcal{H}) \end{cases}$

$$|\mathcal{H}| = \binom{n}{2}, \quad e(\mathcal{H}) = \binom{n}{3}$$

- Graph G $(E(G)) \longleftrightarrow V_G \subseteq V(\mathcal{H})$
- Δ-free $F \in \mathcal{F}_n(K_3) \longleftrightarrow$ indep set V_F
- $\mathcal{F}_n(K_3) \longleftrightarrow \mathcal{I}(\mathcal{H})$ family of all indep sets in $\mathcal{H}$
- Container for $F \in \mathcal{F}_n(K_3) \longleftrightarrow$ Containers for indep sets in $\mathcal{H}$

In this language,

Lem Container for $\mathcal{I}(\mathcal{H})$ for $\mathcal{H}$ encoding Δs. $\Rightarrow \exists\ \mathcal{C}$ family of subsets of $V(\mathcal{H})$ s.t.

(i) $\forall\ I \in \mathcal{I}(\mathcal{H})$, $\exists\ C \in \mathcal{C}$ s.t. $I \subseteq C$.

(ii) $|\mathcal{C}| \leq n^{O(n^{3/2})} \Leftarrow$ (few)

(iii) $\forall\ C \in \mathcal{C}$, almost indep set.

(iv) $\forall\ C \in \mathcal{C}$, $|C| \leq \frac{n^2}{4} + o(n^2) \Leftarrow$

Last time Count – max. Δ-free graphs
– max. sum-free sets

General form (informal) BMS/ST

"nice" hypergraph $\Rightarrow$ $\exists$ "small" family of containers for $\mathcal{I}(\mathcal{H})$
(uniform edge-distribution)

(clustering)

Yet, another simple example

Recall Mantel (1906): Δ-free n-vx $G \subseteq K_n \Rightarrow e(G) \le \frac{1}{2} \binom{n}{2}$

thm Frankl - Rödl (86) $G(n,p)$

$p \gg \frac{1}{\sqrt{n}} \Rightarrow$ a.a.s max size Δ-free $G \subseteq G(n,p)$

Optimal

$$\boxed{e(G) \le \left(\frac{1}{2} + o(1)\right) p \binom{n}{2}}$$

$$ex(G(n,p), K_3) = \left(\frac{1}{2} + o(1)\right) p \binom{n}{2}$$

Rmk $p \ll \frac{1}{\sqrt{n}}$, $\mathbb{E}(e(G(n,p))) \approx p \binom{n}{2}$

$$\mathbb{E}(\# \Delta s) = \binom{n}{3} p^3 \approx n^3 p^3 \ll p n^2$$

Deletion $\Rightarrow \exists G \subseteq G(n,p)$ Δ-free

$e(G) \geq (1-o(1))\, p\binom{n}{2}$

Naive approach   1st moment

X_m = # of m-edge Δ-free graphs in $G(n,p)$

$\mathbb{E}(X_m) \to 0$, when $m \geq (\frac{1}{2}+\varepsilon)\, p\binom{n}{2}$

But # m-edge Δ-free graphs

$\geq \binom{n^2/4}{m}$

Union bound

$\Rightarrow \mathbb{E}(X_m) \geq \binom{n^2/4}{m} p^m = \left(\frac{(\frac{e}{2}+o(1))\, p\binom{n}{2}}{m}\right)^m \to \infty$

blows up

Problem   Too wasteful!

2nd Take

Pf ($p \gg \frac{\log n}{\sqrt{n}}$, true via container)

- Recall $\forall$ container $G \in \mathcal{C}$

$$e(G) \le \frac{n^2}{4} + o(n^2)$$

- Take $\alpha = \Omega(1) > 0$

let $m = \left(\frac{1}{2} + \alpha\right) p \binom{n}{2}$

- If $G(n,p) \supseteq F$ m-edge Δ-free

$\exists\, G \in \mathcal{C}$ s.t. $F \subseteq G$

$$\Rightarrow e(G(n,p) \cap G) \ge m$$

$$\mathrm{Bin}(e(G), p) \Rightarrow \mathbb{E}(\;) = e(G) \cdot p \le \left(\frac{n^2}{4} + o(n^2)\right) p = \left(\frac{1}{2} + o(1)\right) p \binom{n}{2}$$

$$\xrightarrow{\text{Chernoff}} \mathbb{P}\left(e(G(n,p) \cap G) \ge m\right) \le e^{-\beta p n^2}$$

$\beta = \beta(\alpha)$

• union bound over all container $\Rightarrow \mathbb{P}\left(\mathrm{ex}(G(n,p), K_3)\right) \geq m$

$$\leq e^{-\beta p n^2} \cdot n^{O(n^{3/2})} \to 0$$

$$p \gg \frac{\log n}{\sqrt{n}}$$

2) 3-unif case

Basic form

$\mathcal{H}$ 3-unif $\begin{cases} d(\mathcal{H}) = d \\ \Delta_1(\mathcal{H}) = O(d) \\ \Delta_2(\mathcal{H}) = O(\sqrt{d}) \end{cases}$ $\Rightarrow$ $\exists\, \mathcal{C}$ containers for $\mathcal{I}(\mathcal{H})$

i.e.

• $\forall I \in \mathcal{I}(\mathcal{H})$ $\exists\, C \in \mathcal{C}$ s.t. $I \subseteq C$

($\exists K > 0$, $\Delta_1(\mathcal{H}) \leq K \cdot d$ indep others)

• $|\mathcal{C}| \leq \binom{|\mathcal{H}|}{O\left(\frac{|\mathcal{H}|}{\sqrt{d}}\right)}$

$\delta = \Omega(1) > 0$

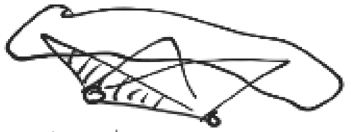
• $\forall C \in \mathcal{C}$ $|C| \leq (1-\delta)|\mathcal{H}|$

iterate

Ex $\mathcal{H}$ (Δ hypergraph $V(\mathcal{H}) = E(K_n)$

$e_1 e_2 e_3 \in E(\mathcal{H})$ $e_1 \Delta e_2$

$$\underline{d(\mathcal{H}) = n-2 = d}$$



$$\Delta_1(\mathcal{H}) = n-2 = O(d(\mathcal{H}))$$

$$\underline{|\mathcal{H}| = \binom{n}{2} \approx n^2/2}$$

$$\Delta_2(\mathcal{H}) \leq 1$$

$$\underline{|\mathcal{C}| = n^{O(n^{3/2})}}$$

$$\sqrt{d} \approx \sqrt{n}$$

$$|\mathcal{H}|/\sqrt{d} \approx n^{3/2}$$

<u>Rmk</u> 1) Final form, $\forall C \in \mathcal{C}$ is almost indep., to get that, we iterate the basic form, i.e. if $\mathcal{H}[C]$ still has lots of edges, apply Basic form to $\mathcal{H}[C]$ $\quad q = |S|$

2) <u>Key idea</u> Input $\underline{I \in \mathcal{I}(\mathcal{H})}$

$I \Rightarrow$ small $\underline{S} \Rightarrow C$

$|S| \leq O(|\mathcal{H}|$

$\sqrt{d}$

fingerprint

containers $\boxed{C = C(S)}$

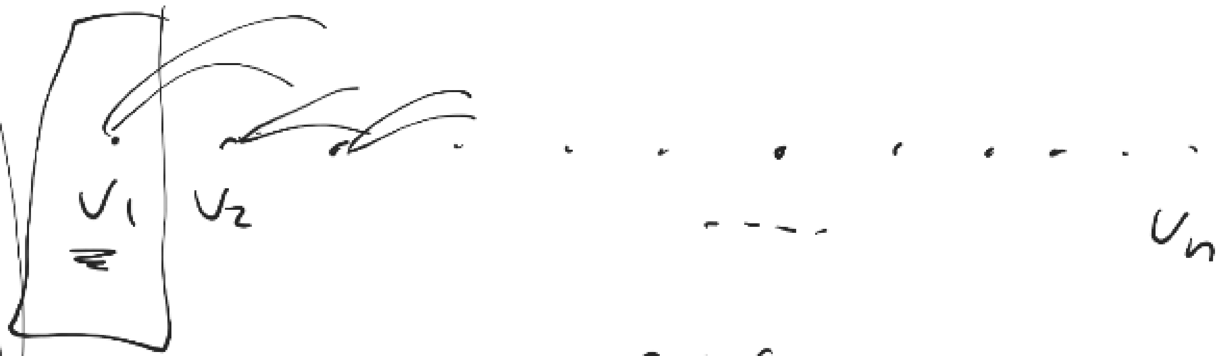
$=$ # fingerprints $\leq \binom{|H|}{O\left(\frac{|H|}{\sqrt{d}}\right)}$

Graph container

Will see KW Alg

Roughly (Graph container) G

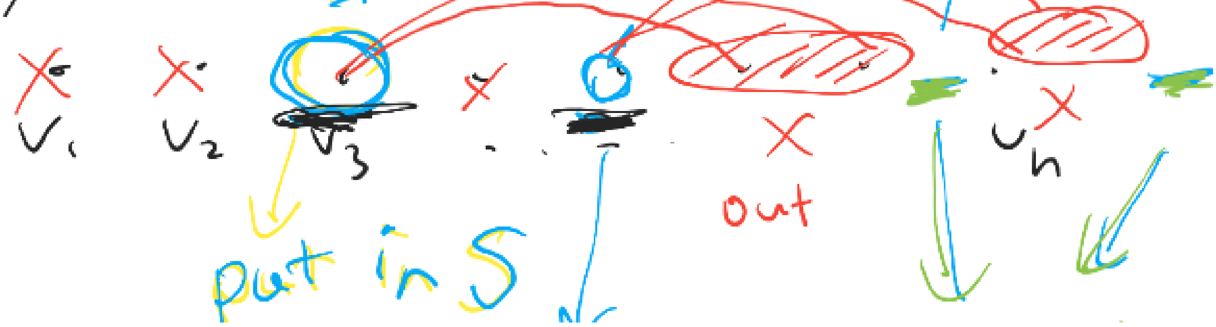
1) Take max-deg ordering



Take $I \in \mathcal{I}(G)$

container $I \cup S$

2) $S \subseteq I$



(q small)

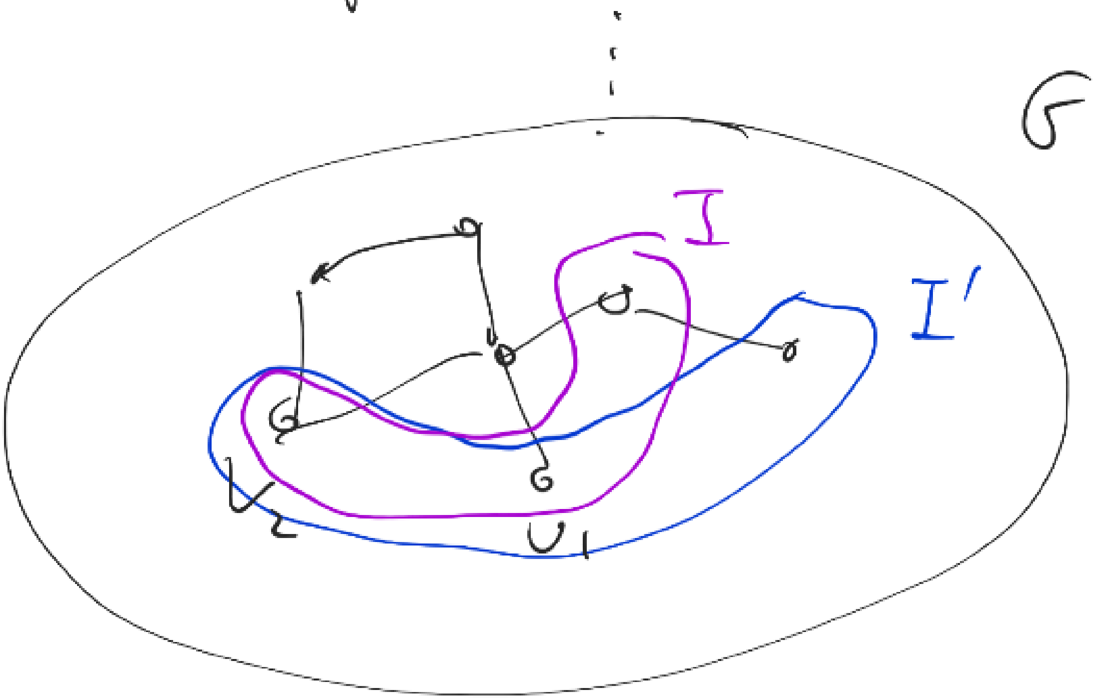
$$\boxed{\#\, q\text{-set} = \binom{|H|}{q}}$$

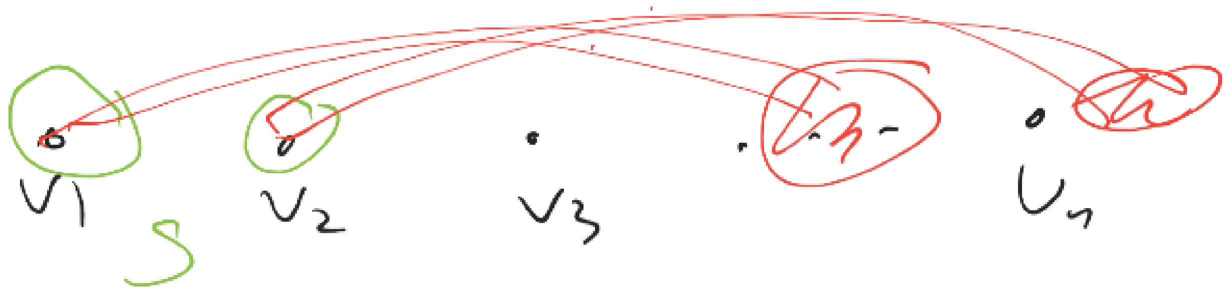
S

remaining vxs depends only S

Next time

- Finish KW Alg.
- Applications of LL.





$\longrightarrow$ 2 steps

$$\mathcal{H}$$

$\Downarrow$ Basic form

$$\mathcal{C}^1 = \{ C^1_1,\ C^1_2,\ \dots,\ C^1_t \}$$

almost indep

too many edges

$$\mathcal{H}[C^1_2]$$

$\Downarrow$ Basic

$$\mathcal{C}^{1,2} = \{ C^{1,2}_{1},\ C^{1,2}_{2}, \dots \}$$

$$\Downarrow$$