Last time

- Hypergraph container for 3-unif

Importance of clustering

(few containers for indep sets in $\mathcal{H}$)

"nice edge distribution"

Random Mantel

$$p \gg \log n / \sqrt{n} \Rightarrow \text{whp } ex(G(n,p), \Delta) = \frac{1}{2} p \binom{n}{2} + o(pn^2)$$

Today

1) KW Alg graph containers

- count # C_4-free graphs
- multiplicative Sidon set
  No $xy = uv$
- supersaturation
  expander mixing
  - count intersecting hypergraphs
  -

Alon-Rödl Sharp lower bound constr. for multi. colour Ramsey.

$|U| \ge R$

1) $\boxed{\text{Lem 1}}$ (Container for graphs)

- G n-vx
- $q \in \mathbb{N}$
- $R > 0$
- $\beta \in [0,1]$

$$\left\{\begin{array}{l} \bullet\ R \ge e^{-\beta q} \cdot n \\ \bullet\ \forall U \subseteq V(G) \\ \quad |U| \ge R \\ \quad \Rightarrow e(G[U]) \ge \beta \binom{|U|}{2} \end{array}\right. \Longrightarrow i(G,m) \le \binom{n}{q}\binom{R}{m-q}$$

supersaturation

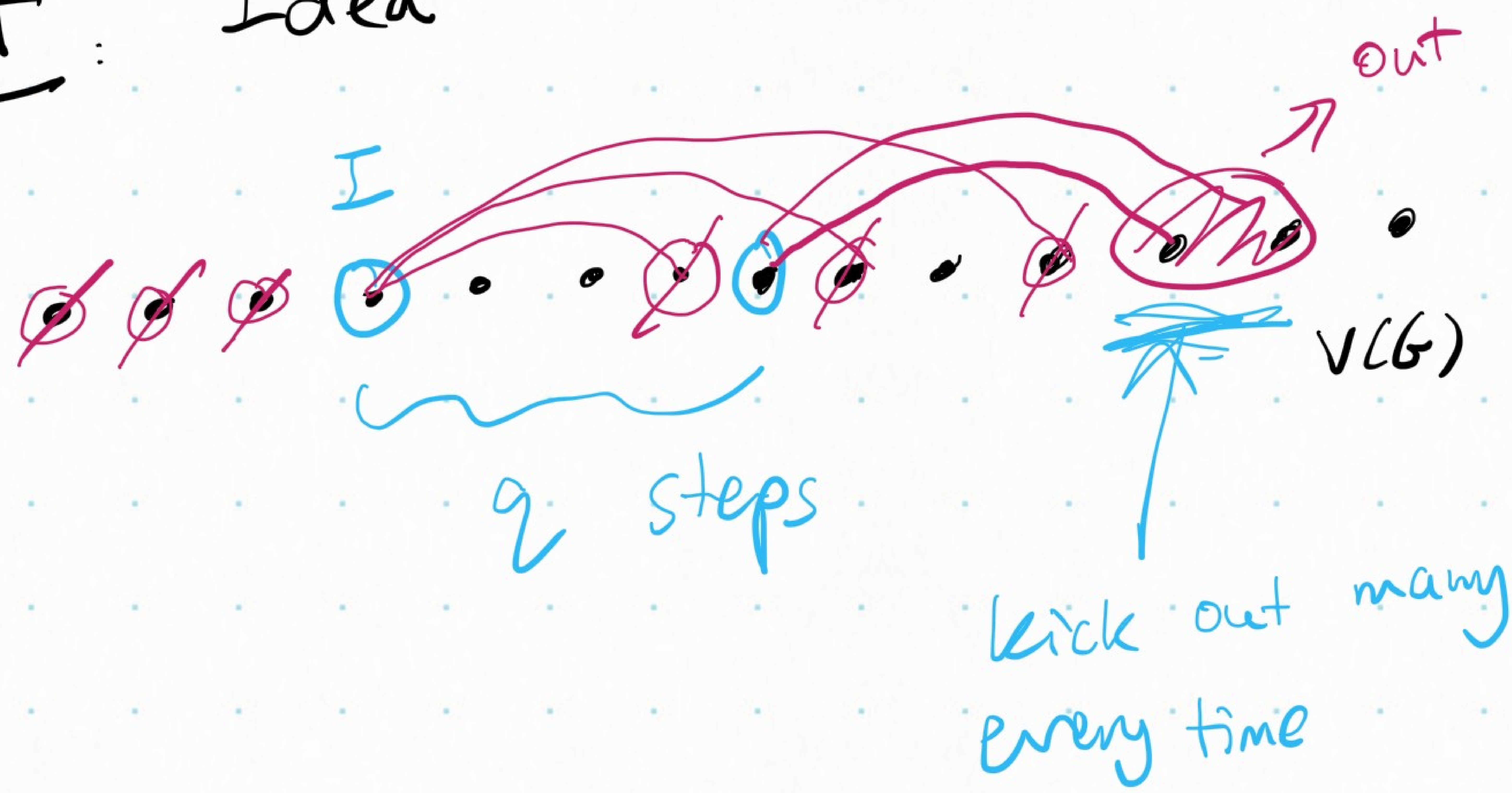
$\forall m \ge q$

Lem 1* Same hypothesis $\boxed{i(G,m) \le \binom{n}{q}\binom{R}{m-q}}$

$\Rightarrow \exists \mathcal{C} \subseteq 2^{V(G)}, \forall C \in \mathcal{C}, \begin{cases} |C| \le R+q \\ |\mathcal{C}| \le \binom{n}{q} \end{cases}$

containers for all indep sets in G

Pf: Idea Fix one $I \in \mathcal{I}(G)$



Details: Order $V(G)$ in max-deg ordering

v_1 v_2 . . . .

largest deg in $G[\{v_2, \cdots, v_n\}]$

1) max-deg ordering

2) Take 1st vx in this ordering that is in I, add it to S.

$A = V(G)$ (initially) active set

$S = \emptyset$ selected

3) delete from A $\begin{cases} \text{preceding vxs} \\ \text{neighbours of selected one} \end{cases}$

update A.

Do this q steps

Output $(v_{n_1}, v_{n_2}, \dots, v_{n_q}) = S$

$A = A(S)$ not depending on I.

Claim $|A| \le R$

Pf (Claim $\Rightarrow$ ☺)

- $S \subseteq I$
- $I \setminus S \subseteq A$
- $A = A(S)$

Note $C = S \cup A \supseteq I$

$$\begin{cases} \# C = |\mathcal{C}| \le \binom{n}{q} \\ |C| \le R + q \end{cases}$$

$$i(G, m) \le \binom{n}{q}\binom{R}{m-q} \quad \blacksquare$$

Pf (Claim) Suppose $|A| > R$

$\Rightarrow e(G[A]) \ge \beta\binom{|A|}{2}$

$\Rightarrow$ at every of the q steps, $d(v_{h_i}) \ge \beta|A|$

$\Rightarrow$ At every step, A shrinks by a factor of $(1-\beta)$

$$|A| \le (1-\beta)^q n \le e^{-\beta q} \cdot n < R$$

Application 1

Q: (Erdős) How many C_4-free graphs on $[n]$?

Known $ex(n, C_4) = \underline{\left(\frac{1}{2} + o(1)\right) n^{3/2}}$

$$2^{cn^{3/2}} \le f_n(C_4) = |\mathcal{F}_n(C_4)|$$

$$\le \sum_{i=0}^{10n^{3/2}} \binom{\binom{n}{2}}{i} \cdots\cdots \le 2^{cn^{3/2}\underline{\log n}}$$

not necessary

Kleitman-Winston 1982

$$f_n(C_4) = 2^{\Theta(n^{3/2})}$$

Rmk: 1) const in the power: open

2) $\log_2(f_n(C_6)) \geq 1.0007\, ex(n, C_6)$

Similar additive Sidon set

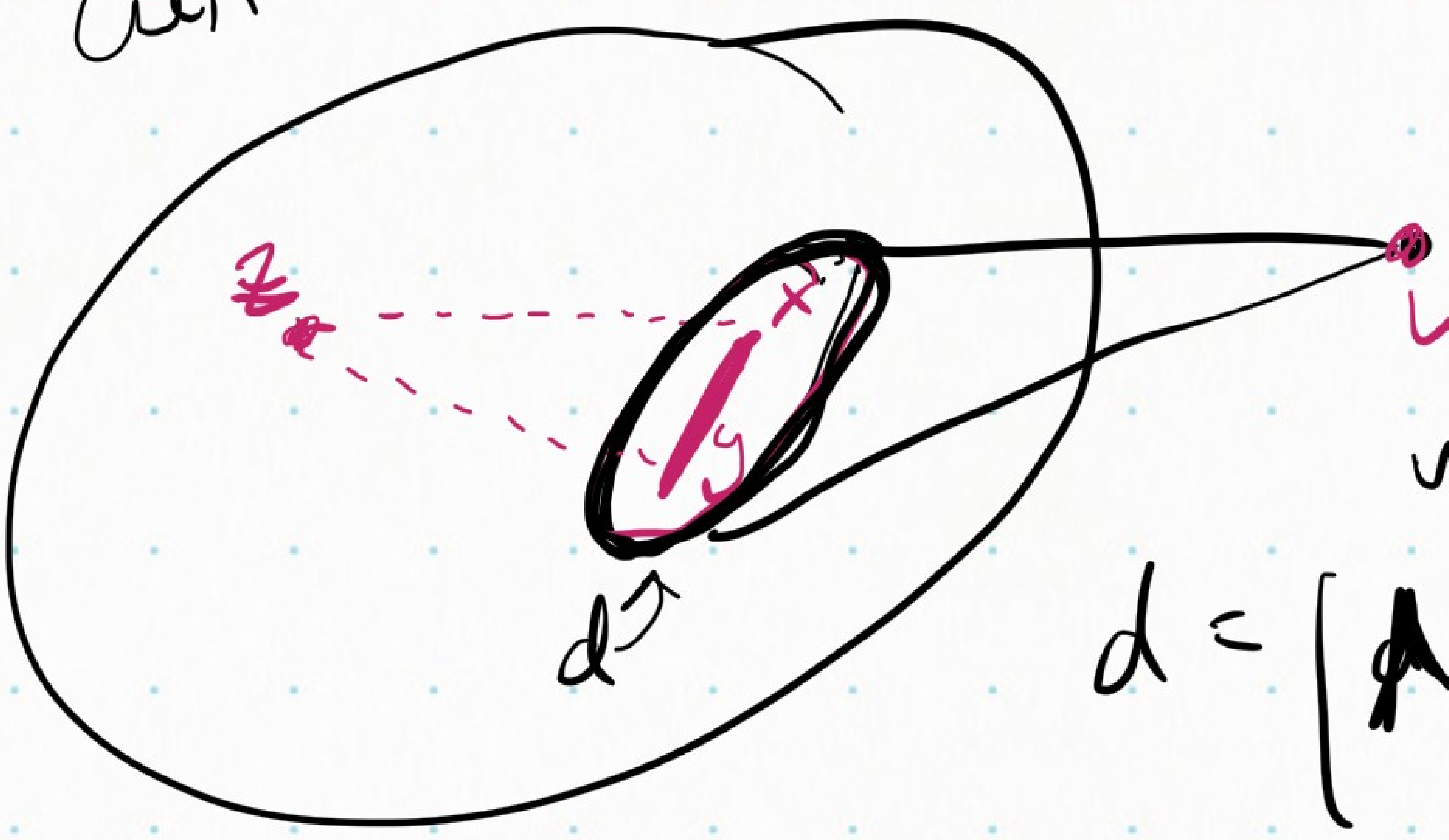
$x+y=u+v$

3) multi. constraints quite different from additive constraints

Idea of the proof

Imagine how we can build one

current G



add one more vx

$v \to G$

$d = |N(v)|$

Consider G^2 $\quad$ $z \sim x$, $\sim y$

adding $v \iff$ picking an indep set in G^2

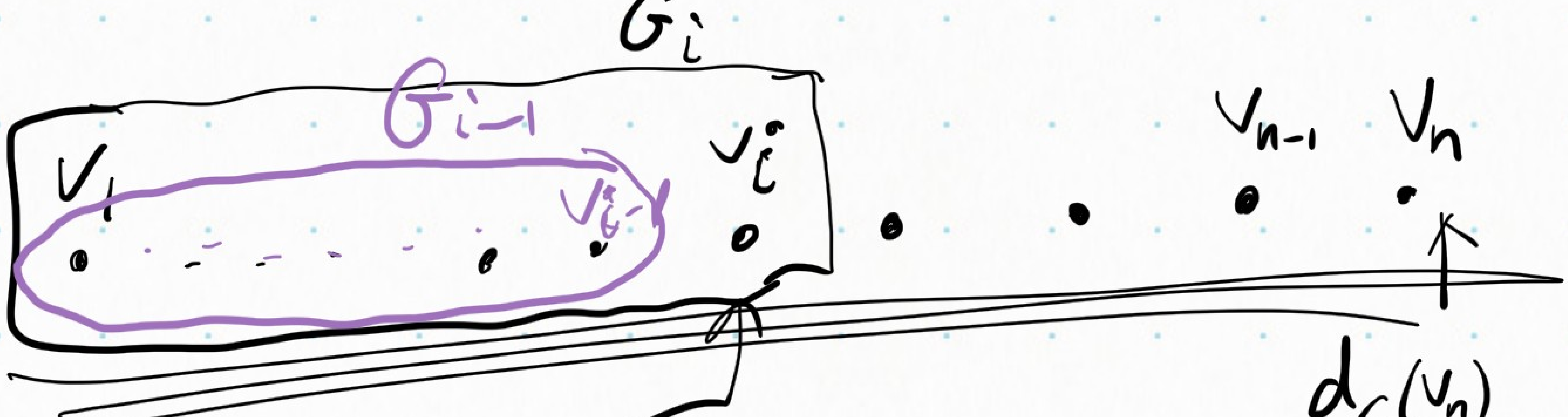
1) d small $\Rightarrow \binom{|G|}{d}$

2) d large $\quad d \geq \sqrt{|G|} / \operatorname{polylog}|G|$

<u>G^2</u> locally dense

Pf (KW C_4 free) Start with C_4-free G $n \to \infty$

- Backward min-deg ordering



$d_{G_i}(v_i) = \delta(G_i)$

$d_G(v_n) = \delta(G)$

$$\Rightarrow \boxed{d_{G_i}(v_i) \leq \delta(G_{i-1}) + 1}$$

- Now consider how a labelled C_4 free graph can be constructed.

1) Choose an ordering (labelled) $v_1, v_2, \ldots, v_n$

2)

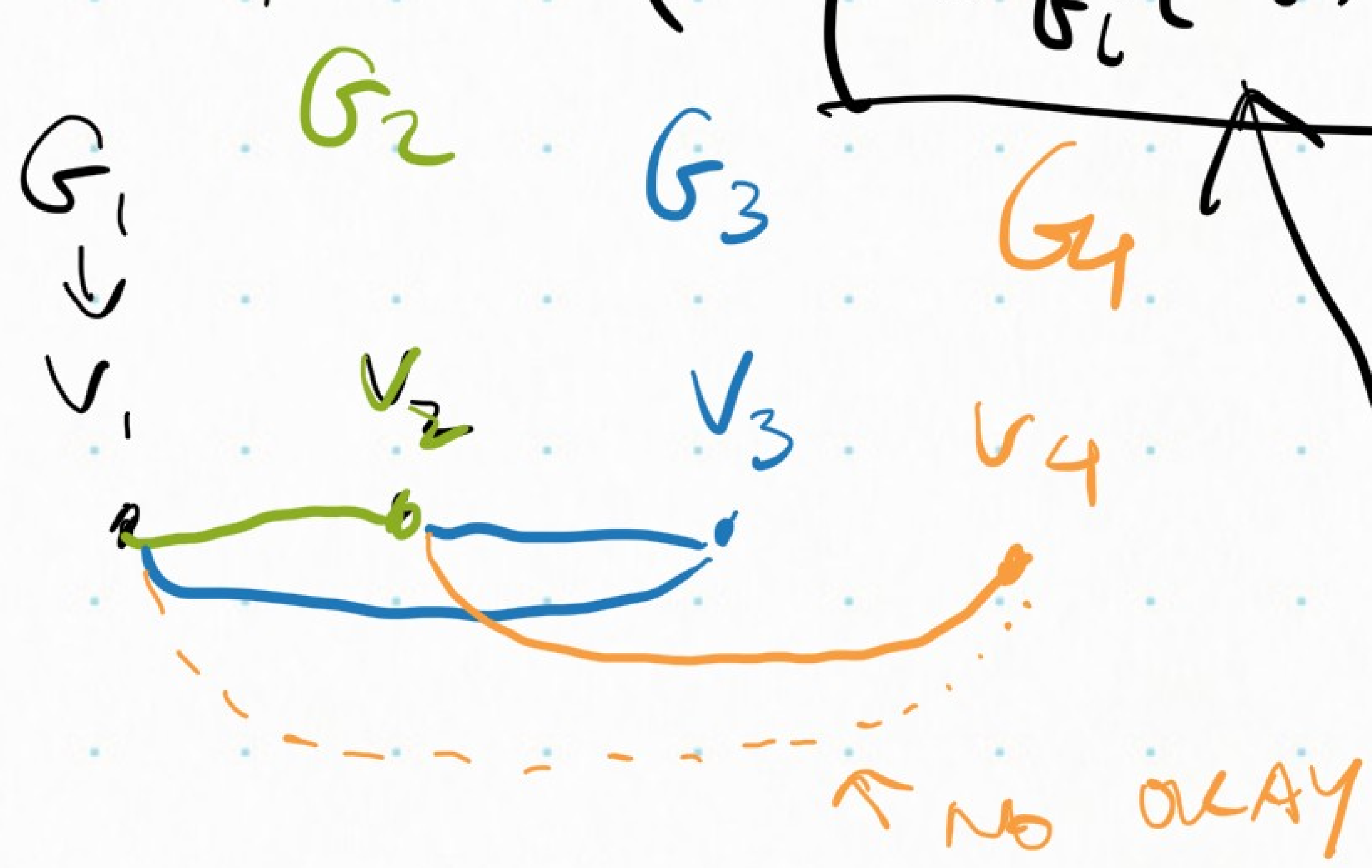
2) - $G_1 = \{v_1\}$

- $\forall\ 2 \le i \le n$

G_i = adding v_i to G_{i-1}

s.t. $\begin{cases} \cdot\ C_4\text{-free} \\ \cdot\ \boxed{d_{G_i}(v_i) \le \delta(G_{i-1})+1} \end{cases}$

d



Def: $\forall\ d \le i-1$

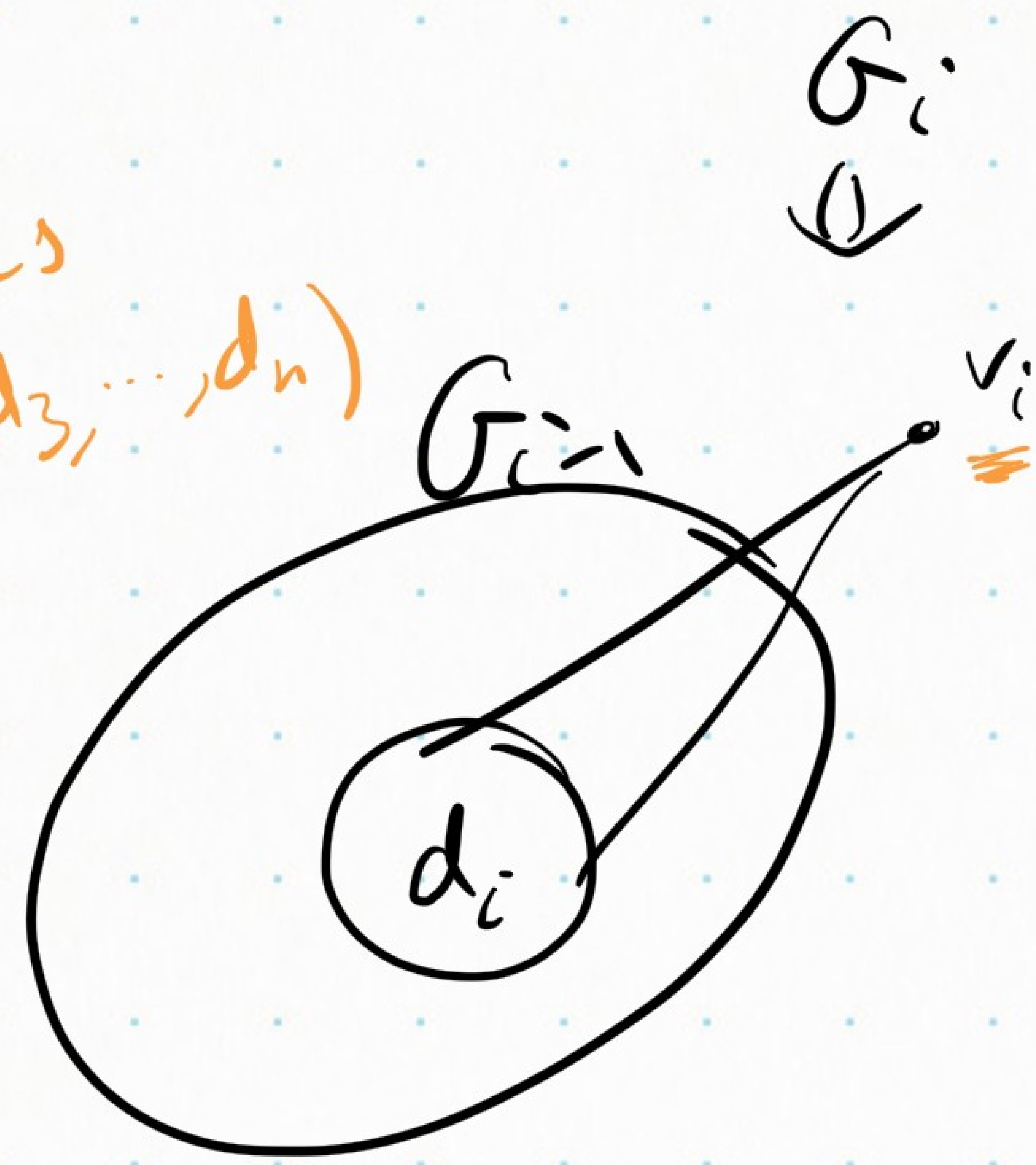
$g_{i-1}(d)$ = max # ways adding a vx of deg d to an $(i-1)$-vx C_4-free with min-deg $d-1$ without creating C_4.

$g_{i-1} = \max\{g_{i-1}(d) : d \le i-1\}$

$$\Rightarrow f_n(C_4) \le \underbrace{n!} \; \underbrace{n!} \prod_{i=2}^{n} g_{i-1}$$

ordering of $v_1, \ldots, v_n$

\# choices for $(d_2, d_3, \ldots, d_n)$



$n! \le 2^{n \log n} \ll 2^{n^{3/2}}$

☆ Suffices to show

$\forall m$,

$g_m \le 2^{c\sqrt{m}}$

for some universal $C > 0$

$d_i = d_{G_{i-1}}(v_i)$

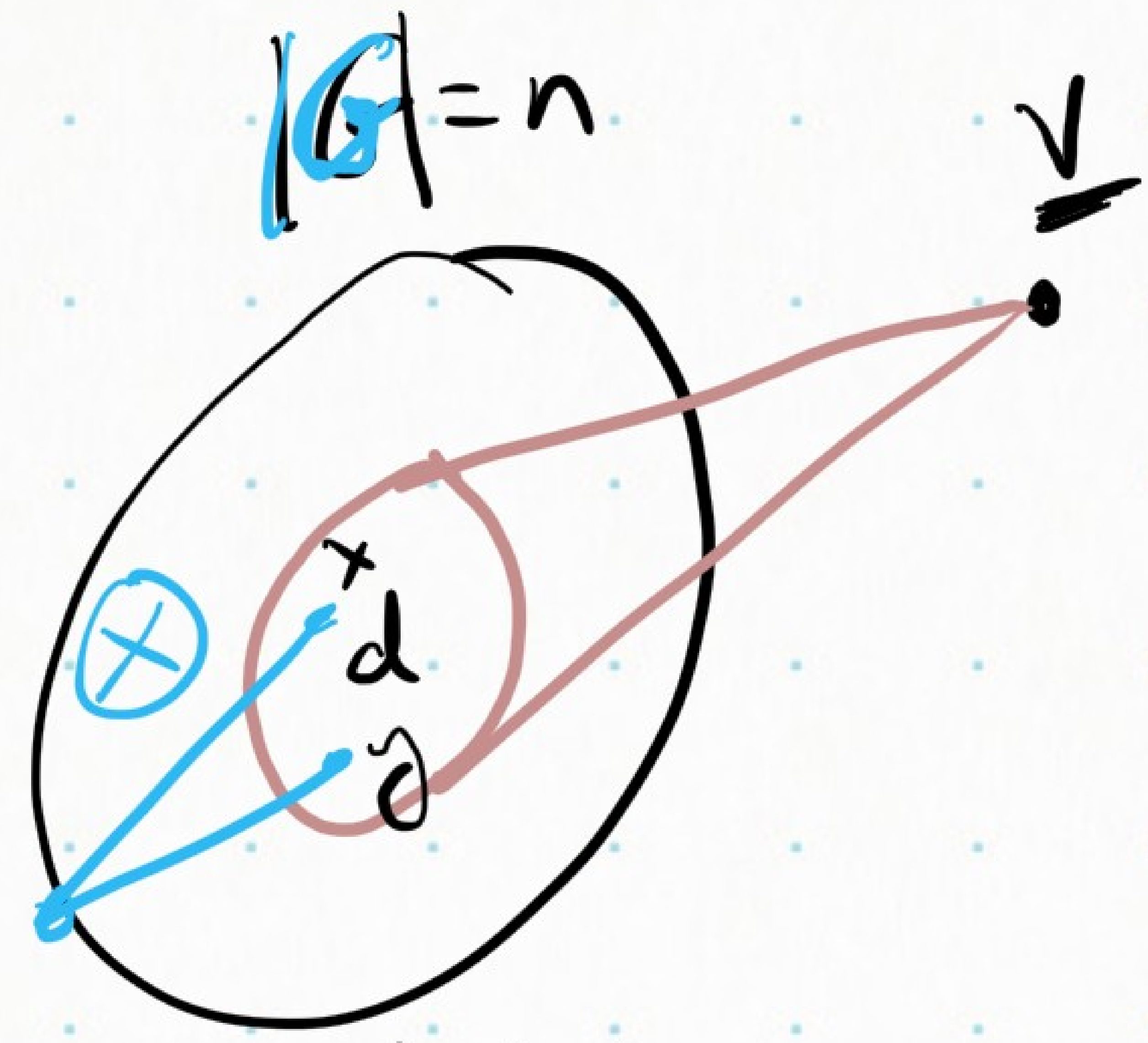
$g_{i-1} \le g_n \le 2^{C \cdot \sqrt{n}}$

$\prod_{i=2}^{n} g_{i-1} \le 2^{C\sqrt{n}\cdot(n-1)} = 2^{Cn^{3/2}}$

Recall $g_n = \max\{g_n(d) : d \le n\}$

\# ways to add deg-d vx to n-vx C_4-free G with $\delta(G) \ge d-1$.

1) $d \le \sqrt{n}/\log n$



$g_n(d) \le \binom{n}{d}$

$\le \left(\frac{en}{d}\right)^d = e^{d \log\left(\frac{en}{d}\right)}$

$\cdots \le 2^{C\sqrt{n}}$

$\delta(G) \ge d-1$

C_4-free

$\binom{a}{b} \le \left(\frac{e \cdot a}{b}\right)^b$

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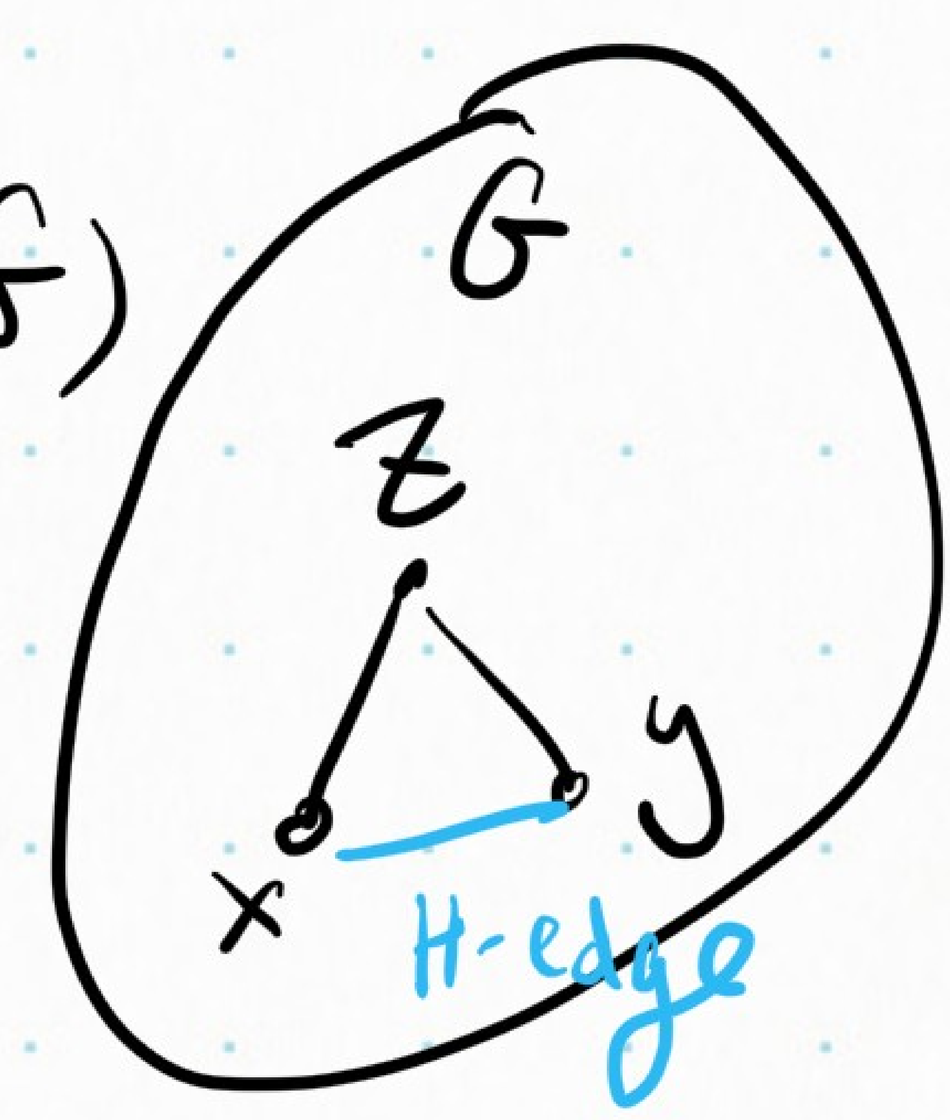
2) $d \ge \sqrt{n}/\log n$

<u>Def</u> $G^2 = H$: $V(H) = V(G)$

$x \underset{H}{\sim} y$ if



Adding v to G without creating C_4

$\Rightarrow$ $N(v)$ has to be an indep set in H

$g_n(d) \le \underset{\sim}{i(H,d)}$

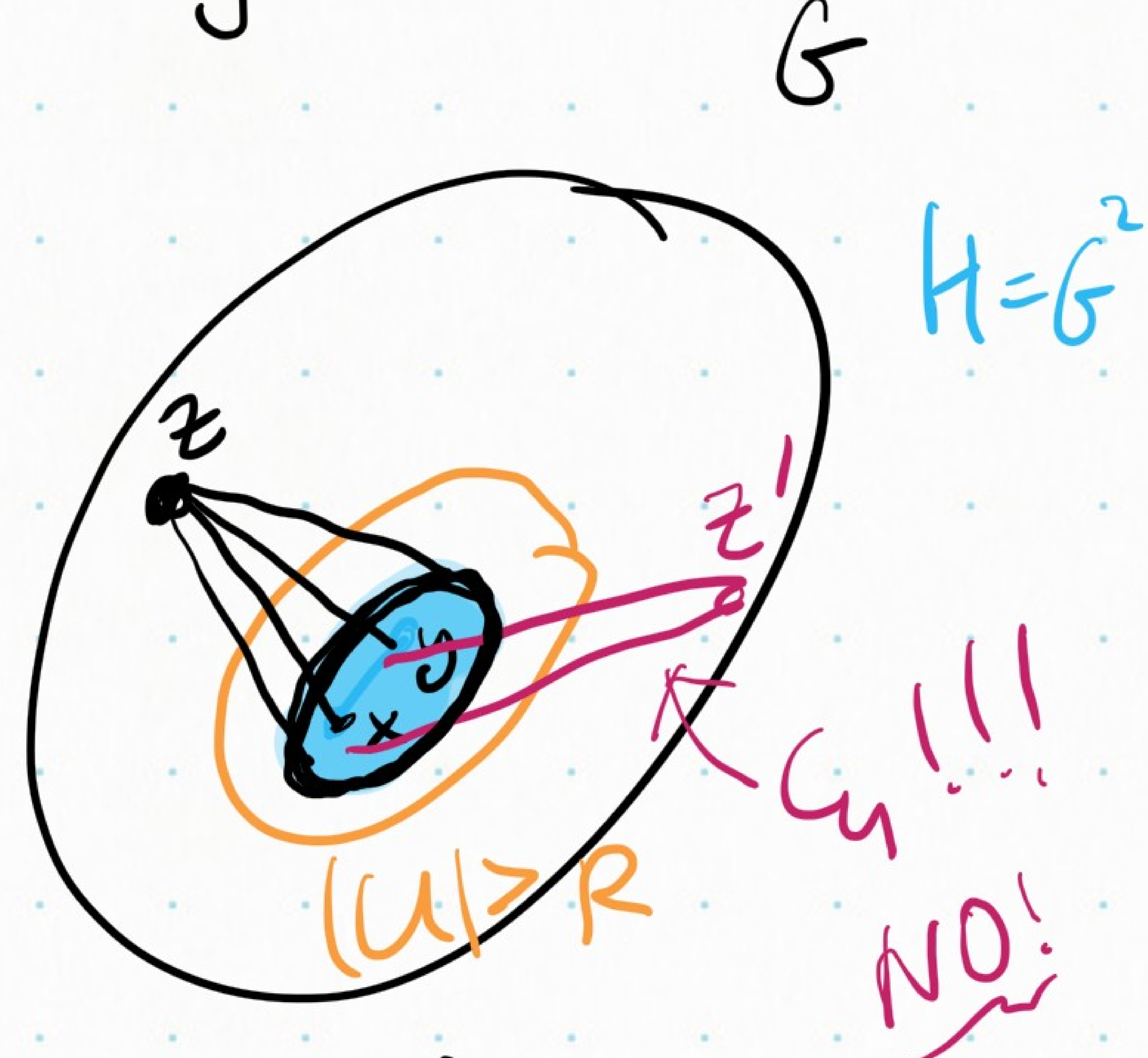
Now left to show

$$i(H,d) \le 2^{C\sqrt{n}}$$

- $H=G^2$ is locally dense

Need

$$e(H[U]) \ge \beta\binom{|U|}{2}$$



G is C_4-free

every edge $xy \in E(H)$

$\longleftrightarrow$ a unique $z \in G$

$$\Rightarrow e(H[U]) = \sum_{z \in V(G)} \binom{d_G(z,U)}{2}$$

convexity

$$\ge n \binom{\dfrac{\sum_{z\in V(G)} d_G(z,U)}{n}}{2}$$

$\approx \frac{d|U|}{n}$

$$\frac{d^2|U|^2}{2n} = \frac{d^2}{n}\binom{|U|}{2}$$

$\sum d(z,U)$

$= \sum_{u \in U} d_G(u)$

$\ge |U|\delta(G) \approx d|U|$

Set $\cdot\ R = |U| \simeq \dfrac{2n}{d}$ $\qquad d \geq \sqrt{n}/\log n$

$\cdot\ \beta = \dfrac{d^2}{n} \geq \dfrac{1}{(\log n)^2}$

$\cdot\ q = (\log n)^3$

$\beta q \geq \log n$

$$\boxed{n \cdot e^{-\beta q} < 1 \leq R}$$

KW graph container

$n^q \leq 2^{(\log n)^4} \longrightarrow$ don't care

Lem 1 $\Longrightarrow$ $i(H,d) \leq \binom{n}{q}\binom{R}{d-q}$ $\qquad \binom{a}{b} \leq \left(\dfrac{ea}{b}\right)^b$

$$\binom{R}{d-q} \leq \left(\frac{e\cdot R}{d-q}\right)^{d-q} \leq \left(\frac{2en}{(d-q)^2}\right)^{d-q}$$

$\dfrac{k}{\sqrt{n}} \cdot 2\sqrt{n}$

$$\leq \left(\frac{e\sqrt{n}}{d-q}\right)^{2(d-q)} \underset{d-q=k}{=} \left(\frac{e\sqrt{n}}{k}\right)^{2k} = \left(\left(e\frac{\sqrt{n}}{k}\right)^{\frac{k}{\sqrt{n}}}\right)^{2\sqrt{n}}$$

$x \left(= \dfrac{k}{\sqrt{n}}\right) > 0 \Rightarrow \left(\dfrac{e}{x}\right)^x \leq e$ $\qquad \leq e^{2\sqrt{n}}$ $\blacksquare$