Last time

- Graph container lemma (KW Alg)
- Count # C_4-free graphs on $[n] = \{1, 2, \cdots, n\}$ $f_n(C_4)$

$$\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$$

$$f_n(C_4) \le 2^{c \cdot n^{3/2}}$$

Application 1

Def A set $S \subseteq [n]$ is multiplicative Sidon if $\nexists$ distinct $a, b, c, d \in S$ satisfying $a \cdot b = c \cdot d$

ex $\{2, 3, 4, 6, 11, 20\}$

NOT multi. Sidon

$$2 \cdot 6 = 3 \cdot 4$$

ex Take $\{\text{primes } p : p \le n\}$

is multi Sidon

$\boxed{\text{No } ab = cd}$

Erdős 38/69

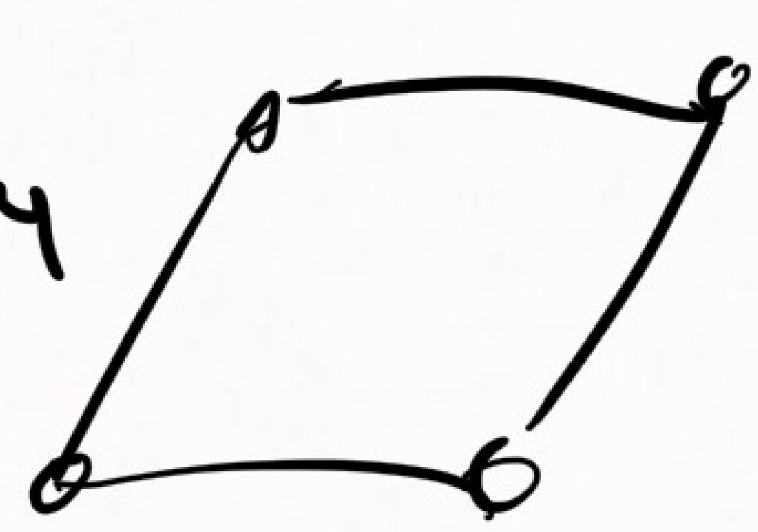
$s(n) =$ max size $S \subseteq [n]$
S is multi Sidon

$$s(n) = \pi(n) + \Theta\left(\frac{n^{3/4}}{\log^{3/2} n}\right) \leftarrow L$$

Lower bound

$\dfrac{\text{primes btw } [n^{2/3}, n]}{\text{auxiliary } G,\ V(G) = \text{primes} \le \sqrt{n}}$

C_4



max-size C_4-free graph

$L = \{u \cdot v : uv \in E(G)\}$

$$|G| = \frac{c\sqrt{n}}{\log n} \Rightarrow e(G) = \frac{c' n^{3/4}}{\log^{3/2} n}$$

Thm L.-Pach 19 $\qquad S(n) = \#$ multi Sidon sets in $[n]$

$$S(n) = T(n) \cdot 2^{\Theta\left(n^{3/4} / \log^{3/2} n\right)}$$

Rmk 1) $T(n) = \prod_{\substack{n^{2/3} < p \le n \\ \text{primes}}} \left(\left\lfloor \frac{n}{p} \right\rfloor + 1 \right)$

$$= 2^{(\alpha + o(1))\pi(n)}, \text{ where}$$

$$\alpha = \sum_{i=1}^{\infty} \frac{1}{i} \log_2\left(1 + \frac{1}{i}\right) \approx 1.8146 > 1$$

2) 1st result up to l.o.t.

$$ex(n, m, C_4) = O(m n^{1/2} + m)$$

3) graph theory kicks when estimating the l.o.t.

Lem (count # unbalanced bipartite C_4-free graphs)

$\forall\, n^{11/12 + o(1)} \le m \le n$

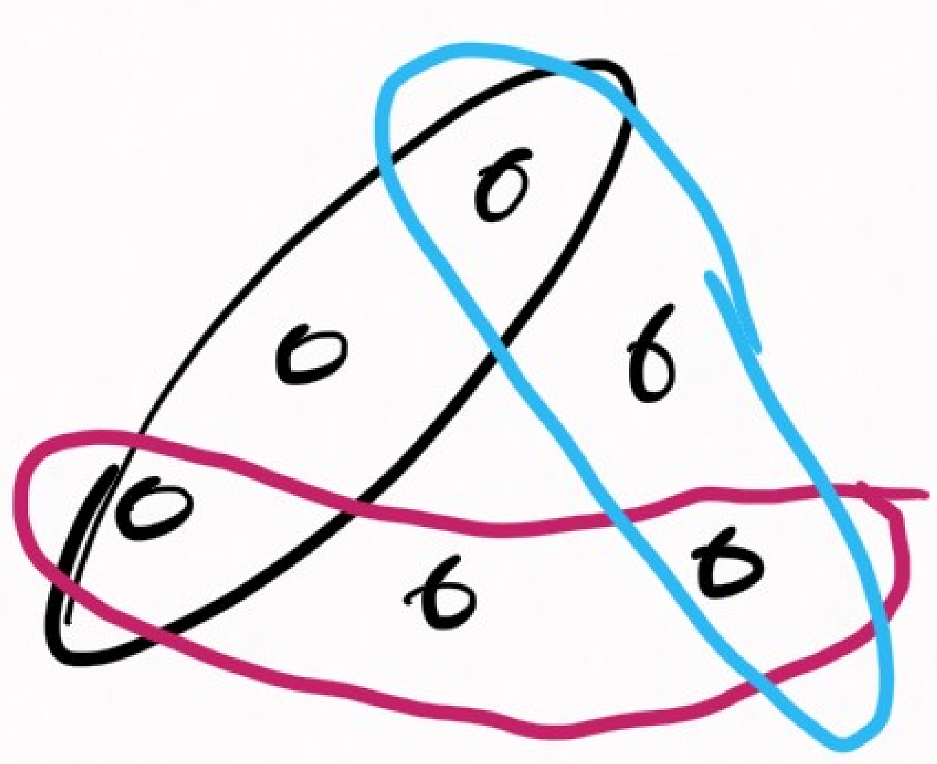
$\#(m, n)$-vx bip. C_4-free graphs $= 2^{O(m n^{1/2})}$

Perhaps $\forall\, m \le n$

$$\#\ (m,n)\text{-vx bip. } C_4\text{-free graphs} = 2^{O(mn^{1/2}+m)}.$$

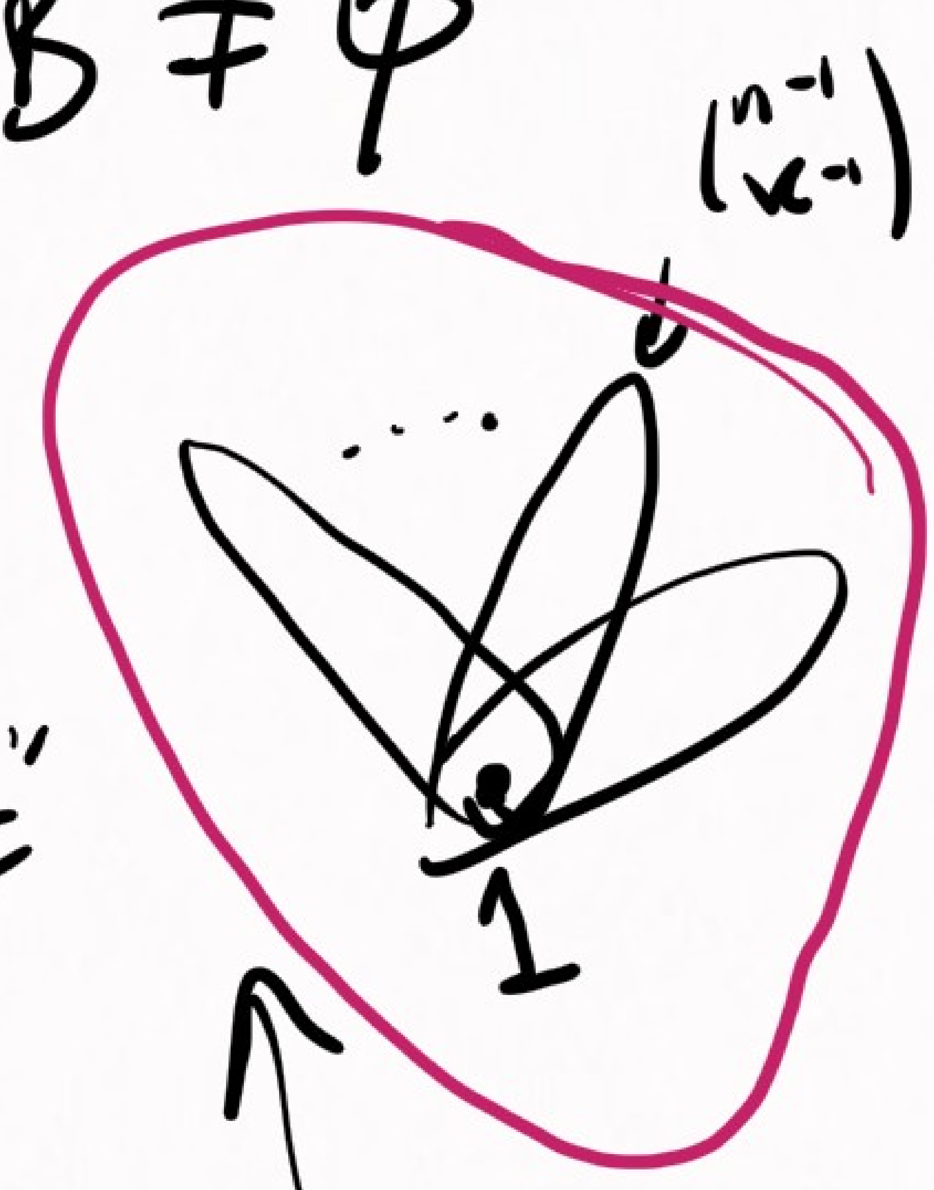
Application 2

Def k-unif hypergraph $\mathcal{H}$ is **intersecting** if $\forall\ A, B \in E(\mathcal{H}),\quad A \cap B \neq \emptyset$



Star

" = "



Erdős-Ko-Rado 61'

$\forall\ n \geq 2k$, k-unif intersecting $\mathcal{H}$ $\Rightarrow e(\mathcal{H}) \leq \binom{n-1}{k-1}$

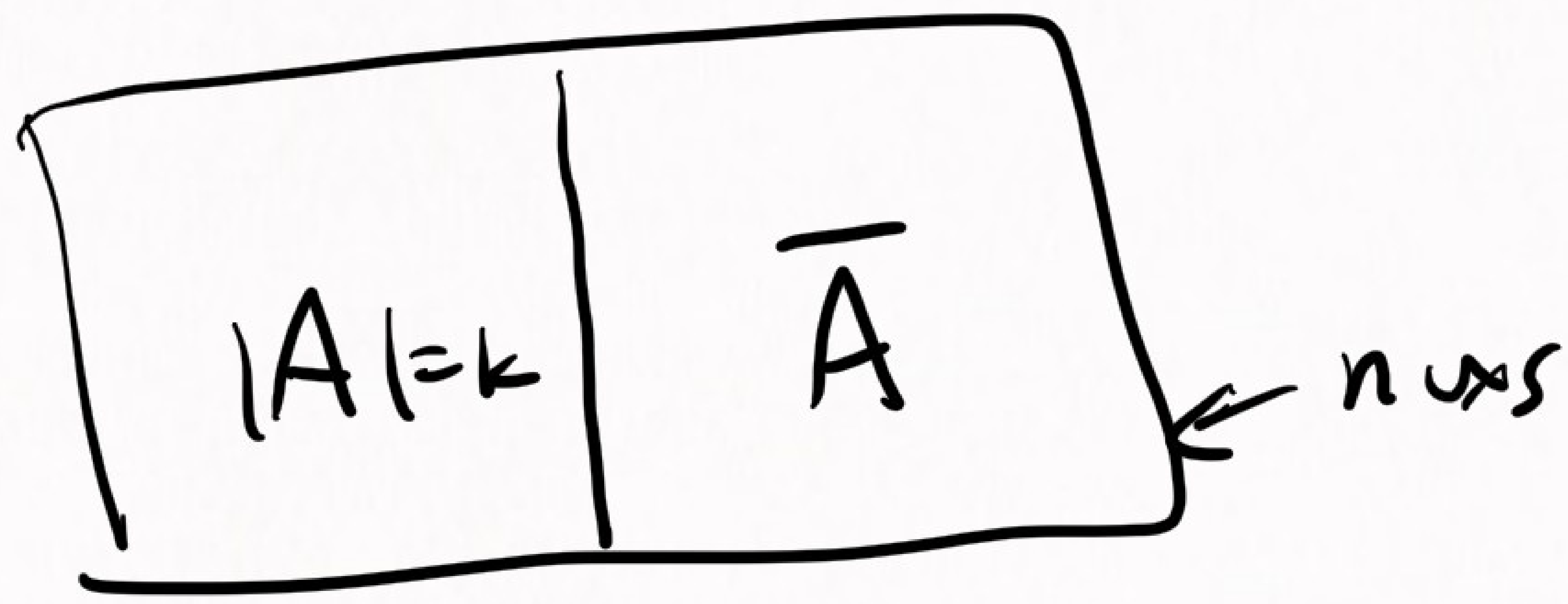
Thm Balogh - Das - Delcourt - L. - Sharifzadeh 2015

$\forall\ n \geq 2k+1$, # k-unif intersecting $= 2^{(1+o(1))\binom{n-1}{k-1}}$.

Rmk $n \geq 2k+1$ tight.

Consider $n = 2k \Rightarrow \underline{\binom{n-1}{k-1}} = \frac{1}{2}\binom{n}{k}$

$\#(A, \bar{A}) = \frac{1}{2}\binom{n}{k}$



For each $(A, \bar{A})$ pair

Choose $\begin{cases} \text{nothing} \\ A \\ \bar{A} \end{cases} \Rightarrow 3^{\frac{1}{2}\binom{n}{k}} = 3^{\binom{n-1}{k-1}}$

Goal Count k-unif intersecting hypergraphs

$\Downarrow$

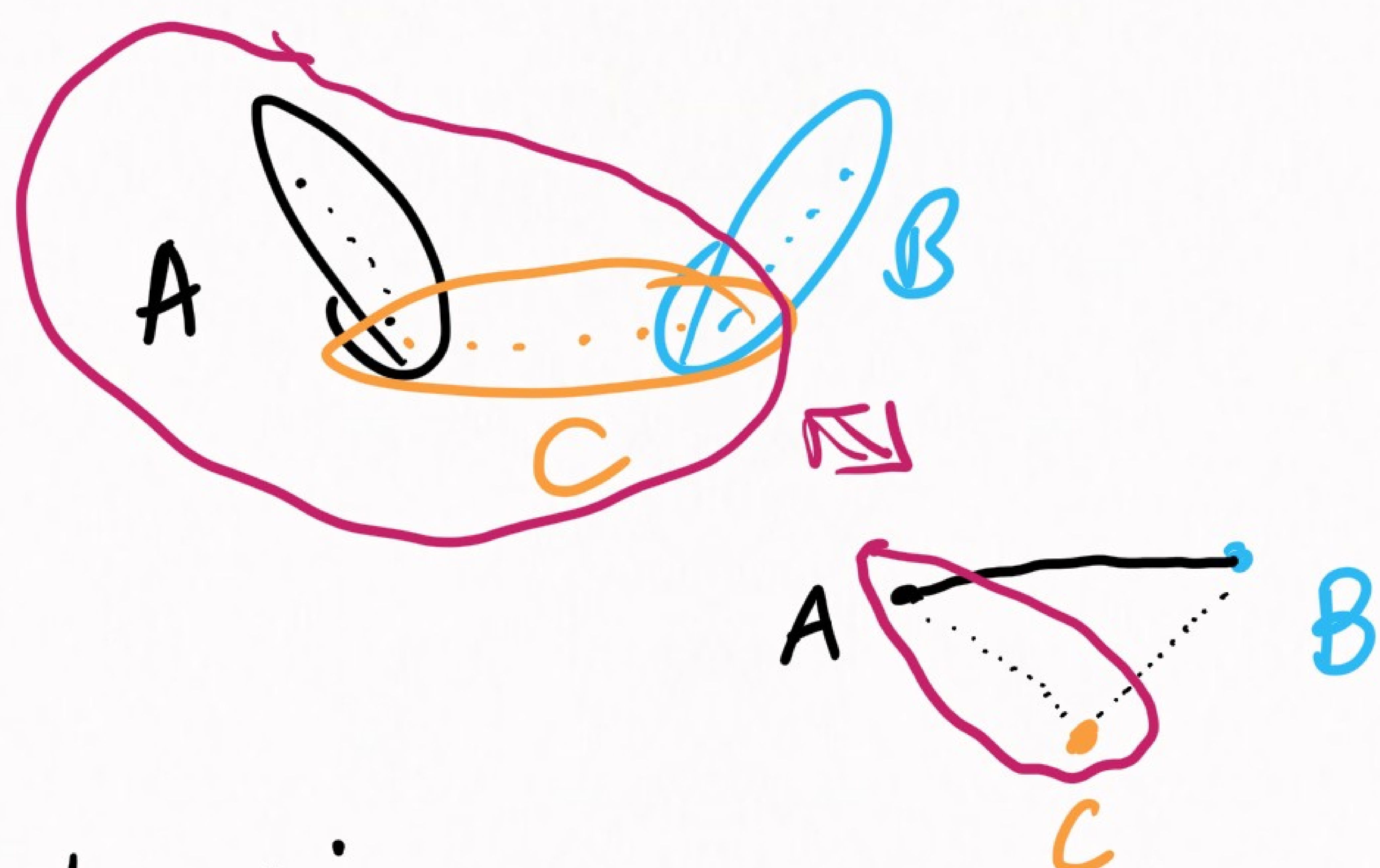
Count indep sets in certain graph

Kneser graph

Def Kneser graph $KG(n,k)$

$$V = \binom{[n]}{k}$$

$$E: \quad A, B \in V, \quad A \sim B \iff A \cap B = \phi$$



k-unif intersecting hypergraph $\mathcal{H}$ $\longleftrightarrow$ an indep set in Kneser graph.

Lem (Graph container)

- n-vx G
- R, β, q

  $R \geq n \cdot e^{-\beta q}$
- locally dense

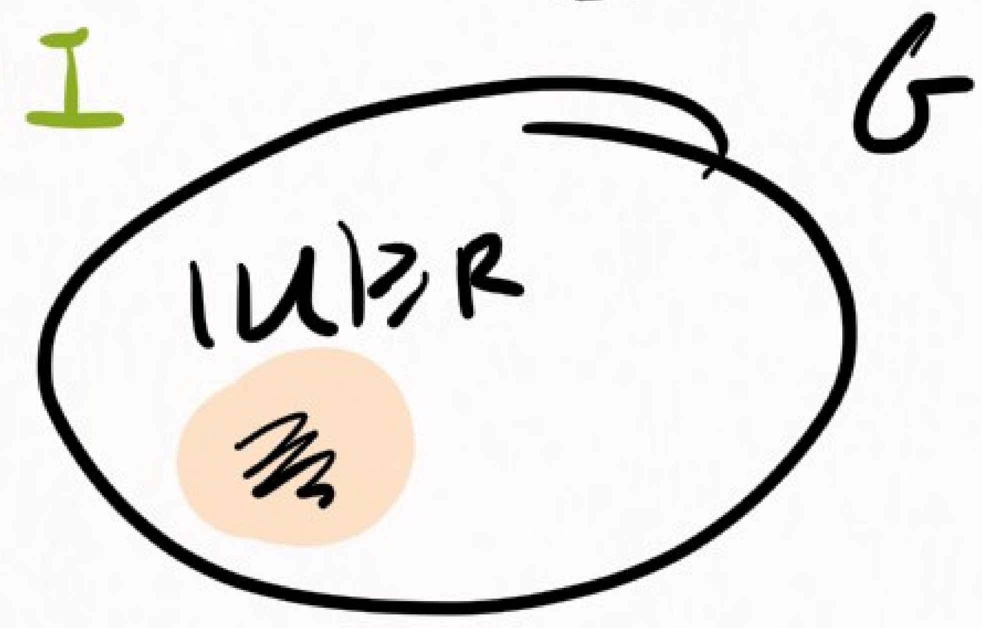
  $\forall U \subseteq V(G), |U| \geq R$

  $e(G[U]) \geq \beta \binom{|U|}{2}$

$\Rightarrow$ $\exists\, \mathcal{C} \subseteq 2^{V(G)}$ s.t.

- $|\mathcal{C}| \leq \binom{n}{q}$
- $\forall C \in \mathcal{C}, \; |C| \leq q + R$
- $\forall$ indep set I

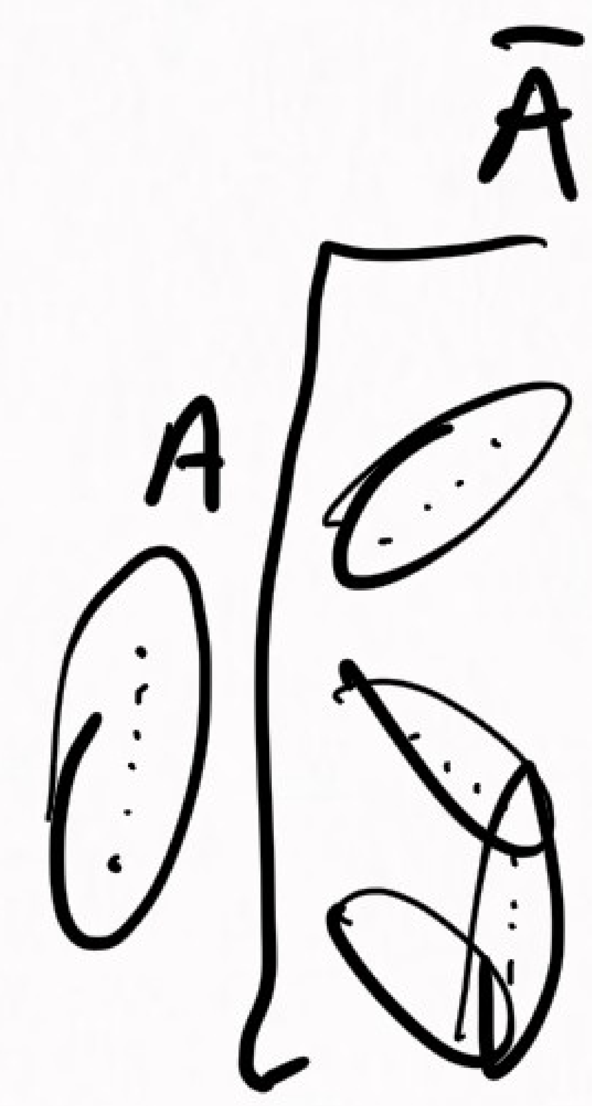
  $\exists\, C \in \mathcal{C}$ s.t. $I \subseteq C$



Lovász 79 $\lambda_{min} = \lambda = -\binom{n-k-1}{k-1}$

Kneser $N = \binom{n}{k}$

D-reg, $D = \binom{n-k}{k}$



Def (n, d, λ)-graph (pseudorandom)

n-vx, d-reg, $\max\{|\lambda_2|, |\lambda_n|\} \le \lambda$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$\lambda << d \Rightarrow$ pseudorandom

Lem (Expander mixing Lem Alon-Chung '88)

G n-vx, d-reg, $\lambda = \lambda_n$ $\Rightarrow$ $\forall A \subseteq V(G)$ edge-density

$$e(G[A]) \geq \frac{d}{n} \cdot \frac{|A|^2}{2} + \frac{\lambda}{2n}|A|(n-|A|)$$

expected to see in random graph

$\leftarrow$ error

Prop (supersaturation)

$\forall \varepsilon > 0$, $\forall\ |S| \geq \underbrace{(1+\varepsilon)\binom{n-1}{k-1}}_{R}$ vxs in Kneser graph $\Rightarrow$ $e(G[S]) \geq \underbrace{\left(1 - \frac{1}{1+\varepsilon}\right)\frac{Dn}{N(n-k)}}_{\beta}\binom{|S|}{2}$

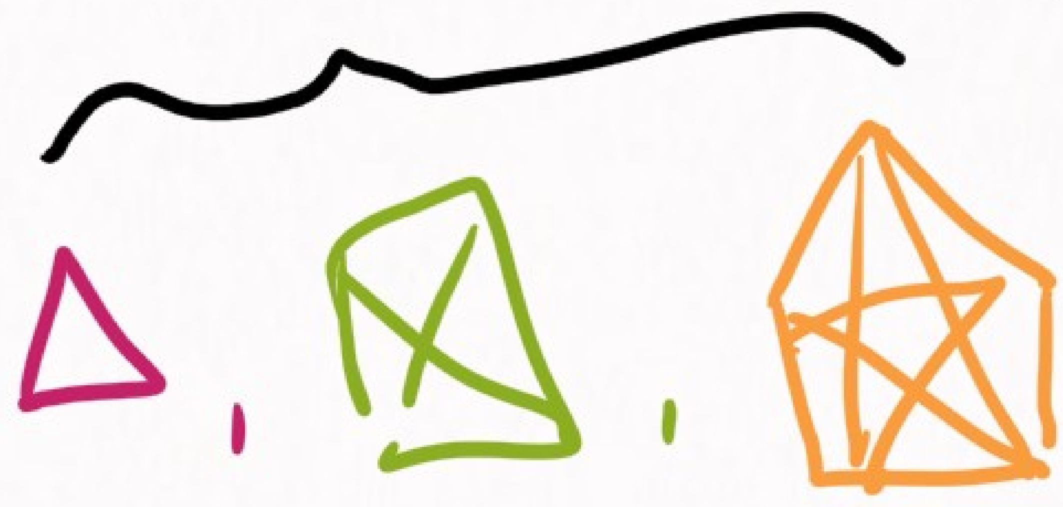
Rmk 1. R almost best we can hope for, as $R \underset{\text{has to be}}{>} \alpha(G) = \binom{n-1}{k-1}$

Application #3 Lower bound constr. in multicolour Ramsey.

Def $R(K_s, K_t) = \min n$ s.t.

$\forall$ 2-edge-colouring of $K_n \Rightarrow \begin{cases} K_s \\ \text{or} \\ K_t \end{cases}$

k colours

$R($  $)$

Recall $R(\Delta, K_m) = \underset{\gtrsim}{\Theta}\left(\frac{m^2}{\log m}\right)$

Open const. $\left[\frac{1}{4}+o(1),\ 1+o(1)\right]$

- Erdős–Sós 79 Conj

$$\lim_{m\to\infty} \frac{R(\Delta, \Delta, K_m)}{R(\Delta, K_m)} = \infty$$

That is, need a graph (construction) on $\gg \frac{m^2}{\log m}$

s.t. $\begin{cases} \alpha(G) < m \\ G \text{ is } (\Delta,\Delta) \text{ free} \end{cases}$

$\exists c_1, c_2$ s.t.

$m^3 \log^{c_2} m \geq \cdot \geq m^3 \log^{c_1} m$

Thm (Alon-Rödl 05)

$$\boxed{\bullet\ R(\Delta, \Delta, K_m) = \Theta(m^3 \cdot \text{polylog}\, m)}$$

- $R(\underbrace{\Delta, \ldots, \Delta}_{k}, K_m) = \Theta(m^{k+1}\ \text{polylog}\, m)$

Goal: To show $R(\Delta, \Delta, K_m) > m^3 \cdot \text{polylog}\, m$

Construct G $\begin{cases} \cdot\ (\Delta,\Delta)\text{-free} \quad E(G) = E(G_1) \cup E(G_2) \quad (\Delta\text{-free}) \\ \cdot\ \alpha(G) < m \approx n^{1/3}\, \text{polylog}\, n \\ \cdot\ |G| = n \approx m^3\, \text{polylog}\, m. \end{cases}$

Pf Idea

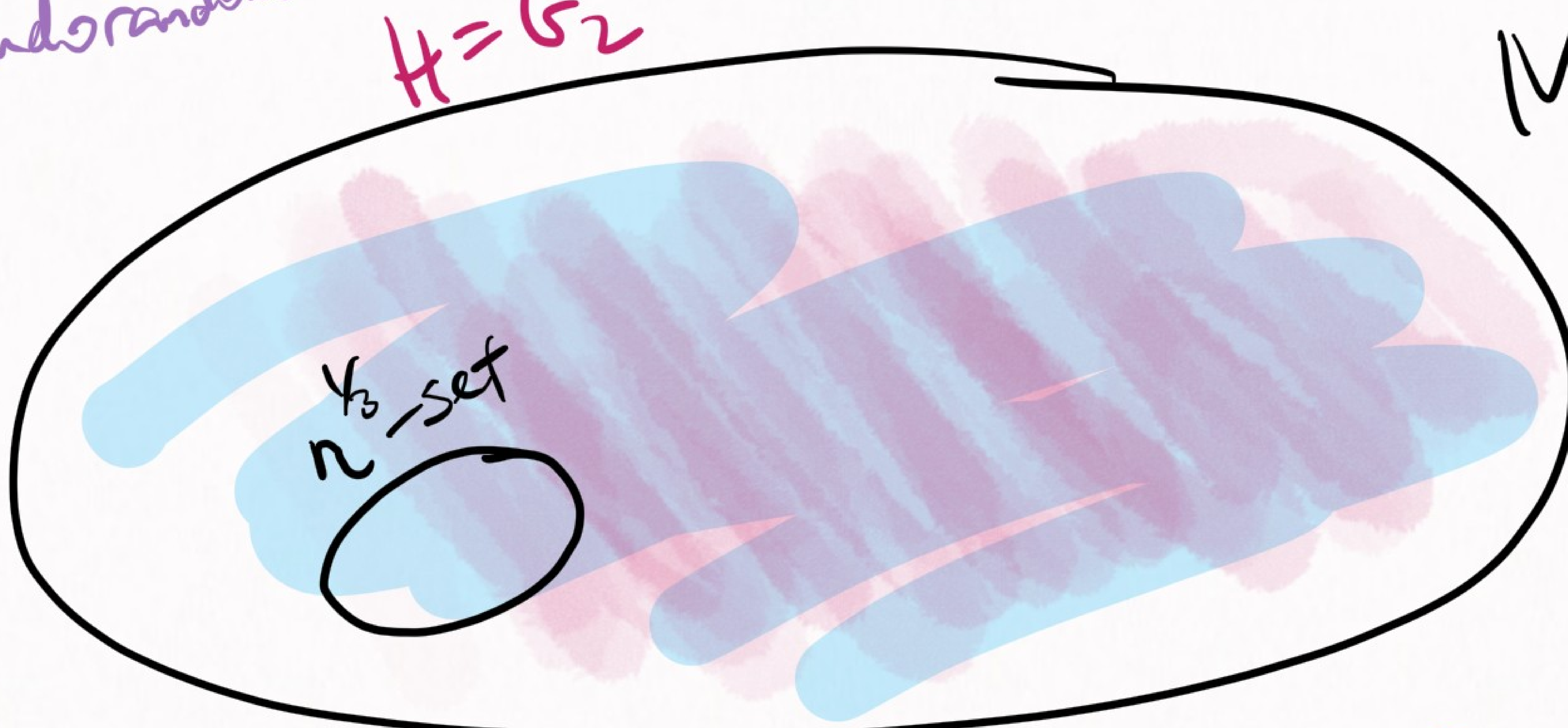
Take H $\begin{cases} n\text{-vx} \\ \Delta\text{-free} \\ \text{"few" indep sets} \gtrsim n^{1/3} \end{cases}$

such H is dense pseudorandom Δ-free

Let $G = G_1 \cup G_2$

$H = G_2$

$|V| = n$



random $V(H) \to V$

$H = G_1$