Lem (AR) n-vx Δ-free G, write $M = i(G,m)$. Suppose $\exists\, k \in \mathbb{N}$, $k \geq 2$ s.t.

$$M^k < \left(\binom{n}{m}\right)^{k-1}.$$

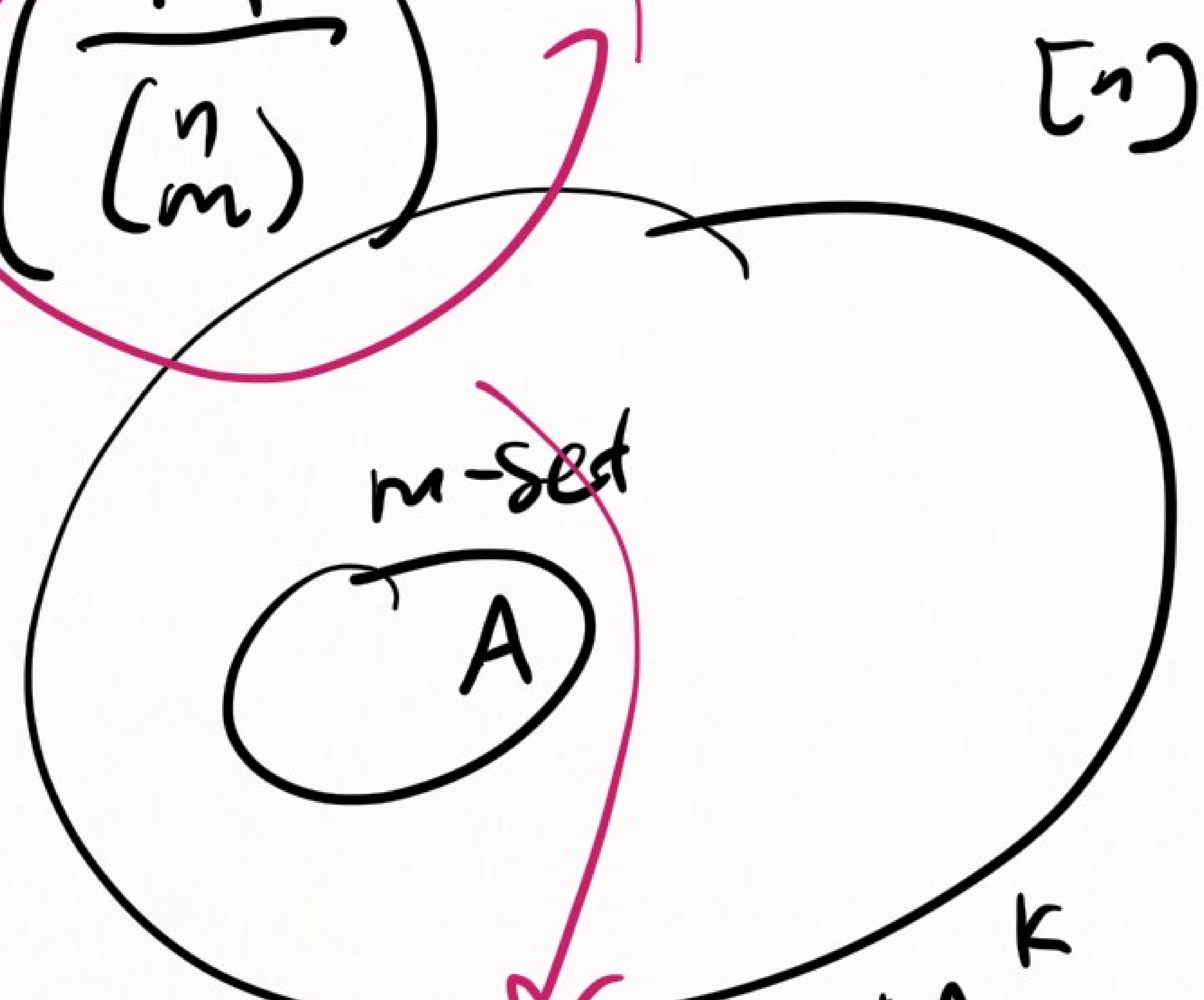
$$\Rightarrow \quad R(\underbrace{\Delta, \dots, \Delta}_{k}, K_m) > n.$$

Pf • For each $i \in [k]$ colours

Take random $\sigma_i : V(G) \to [n]$

• Consider a fixed m-set A in $[n]$

$$\mathbb{P}(A \text{ is indep in all } G_i) \leq \left(\frac{M}{\binom{n}{m}}\right)^k$$



• Union bound

$$\mathbb{P}(\exists \text{ an } m\text{-set indep in } \cup G_i) \leq \binom{n}{m} \cdot \left(\frac{M}{\binom{n}{m}}\right)^k = \frac{M^k}{\binom{n}{m}^{k-1}} < 1$$

$\Rightarrow \exists$ a choice of overlapping of G_i

s.t. $\alpha(\cup G_i) < m$ $\blacksquare$

Consider $k=2$

$$R(\Delta, \Delta, K_m) > \underset{n}{\underbrace{m^3 \text{polylog} m}}$$

<u>By Lem</u> it's sufficient to find H

n-vx

Δ-free

$M = i(H, m)$, $\boxed{M^2 < \binom{n}{m}}$

crude bd on M

$M < \binom{cn}{m}$

Choose dense pseudorandom Δ-free (Alon)

H

(n,d,λ)-graph $\begin{cases} d = \Theta(n^{2/3}) \\ \lambda = \Theta(n^{1/3}) \end{cases}$

$\boxed{\lambda << d}$

(n,d,λ) $\begin{cases} d=\Theta(n^{2/3}) \\ \lambda=\Theta(n^{1/3}) \end{cases}$

H

$\boxed{\begin{array}{l} \underline{\text{Lem}} \quad \forall \varepsilon>0,\ \exists C>0,\ n_0>0 \text{ s.t. } \forall n\geq n_0 \text{ s.t.} \\ \forall\ (n,d,\lambda)\ G \\ \qquad \Rightarrow i(G,m) \leq \dbinom{\left(\frac{\lambda}{d+\lambda}+\varepsilon\right)n}{m} \\ \forall m \geq C\cdot n/d. \end{array}}$

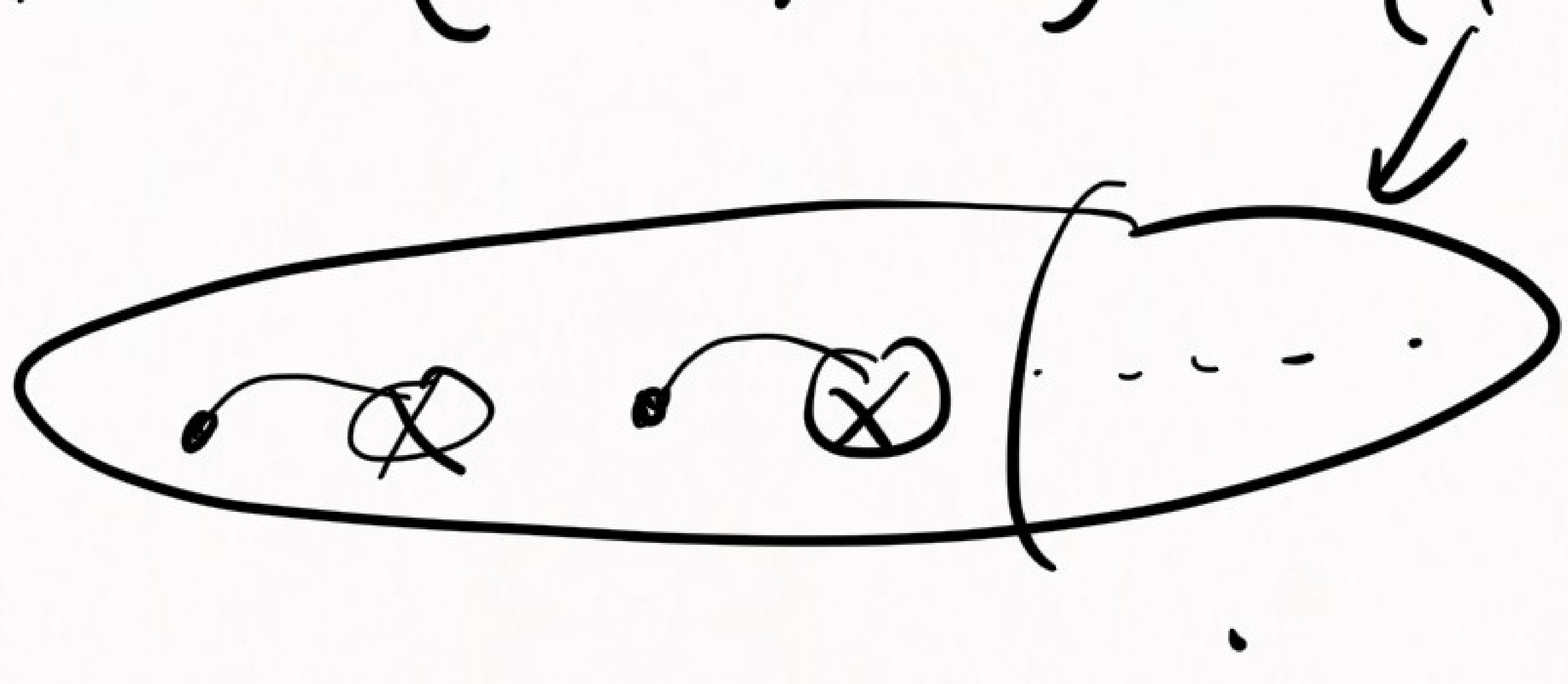
Rmk 1 $\frac{\lambda}{d+\lambda}$ is optimal.

Hoffman bd is tight.

Rmk 2 bd on m is tight up to const. factor. $m=o(n/d)$

$$i(G,m) = \binom{(1-o(1))n}{m}$$

$(1-o(1))n$



Pf Take $|U| \geq R = \left(\frac{\lambda}{d+\lambda} + \frac{\varepsilon}{2}\right) n$

expander mixing $\Longrightarrow$

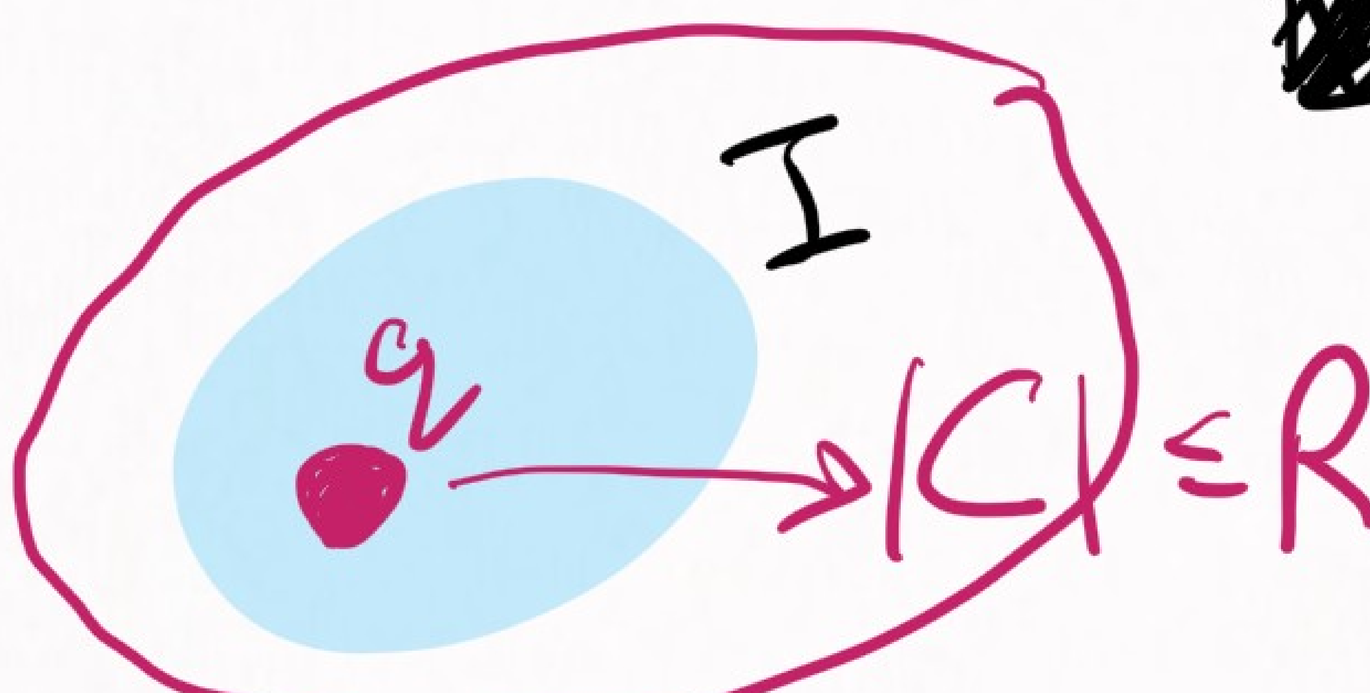
$$e(G[U]) \geq \underbrace{\frac{\varepsilon d}{n}}_{\beta} \binom{|U|}{2}$$

Graph container

locally dense

Choose q s.t. $e^{-\beta q} \cdot n \leq R$

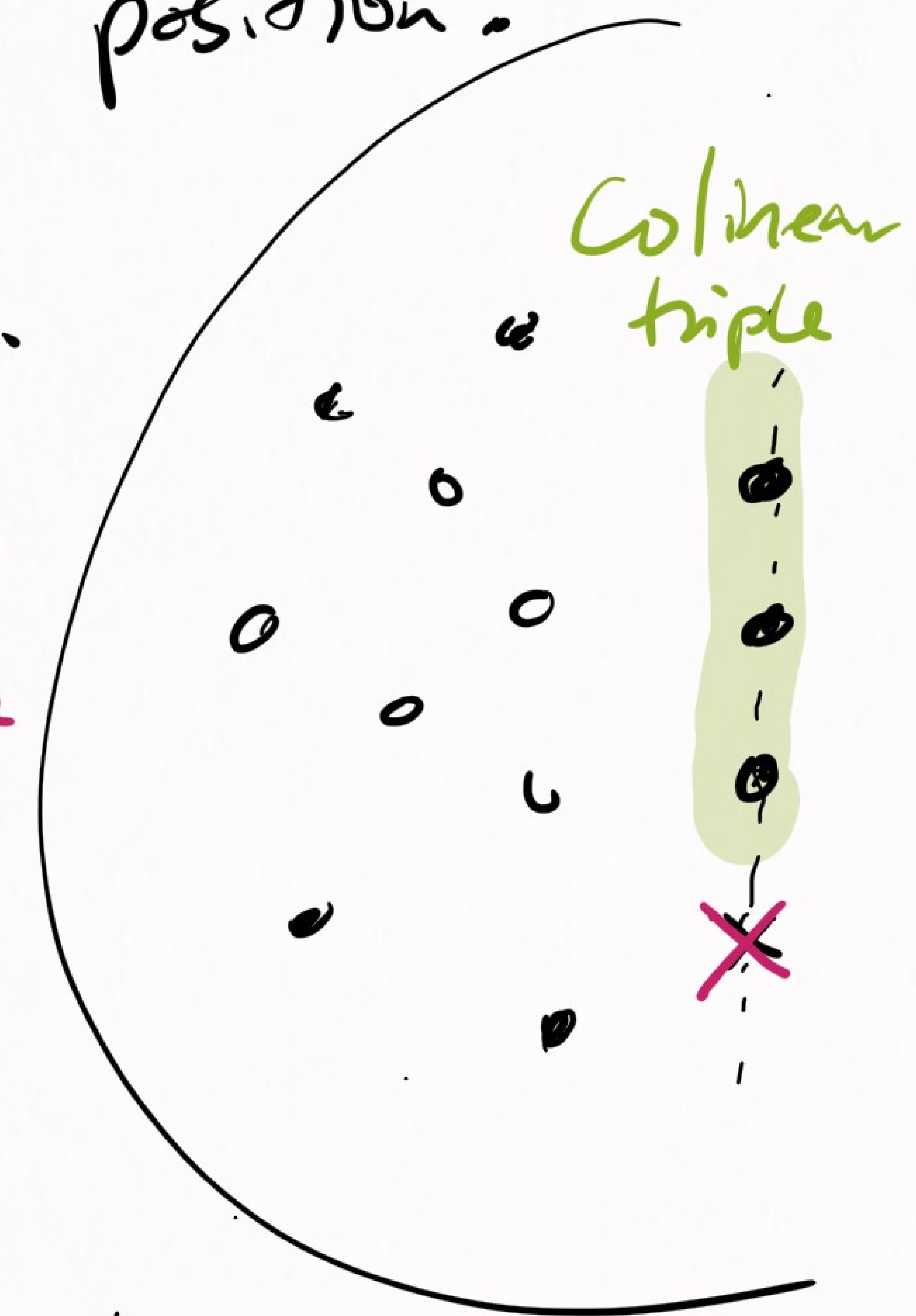
graph contai $\Longrightarrow$ $i(G,m) \leq \binom{n}{q}\binom{R}{m-q} \cdots$



Graph container (KW)

- G, n-vx
- β, q, R with $e^{-\beta q} \cdot n \leq R$ $\Longrightarrow$ $i(G,m) \leq \binom{n}{q}\binom{R}{m-q}$
- locally dense $\forall |U| \geq R$, $e(G[U]) \geq \beta \binom{|U|}{2}$

Q (Erdős) Given n pts on $\mathbb{R}^2$ with no colinear 4-tuple, how large a subset in general position? (no 3 on a line)



Def $S \subseteq \mathbb{R}^2$ with no col. 4-tuple

let $\alpha(S)$ = size of largest $T \subseteq S$ s.t. T is in general position.

$$\alpha(n) = \min_{|S|=n} \alpha(S)$$

upper bound

$\Uparrow$

construction

Füredi 91

$\omega(\sqrt{n \log n}) \le \alpha(n) \le o(n)$

Balogh - Solymosi 19

$$\boxed{\alpha(n) \leq n^{5/6+o(1)}}$$

$\Uparrow$

Need to find a set $|R| = n$

- no col. 4-tuple ✓
- $\forall$ subset of R with size $n^{5/6+o(1)}$ IS NOT in general position

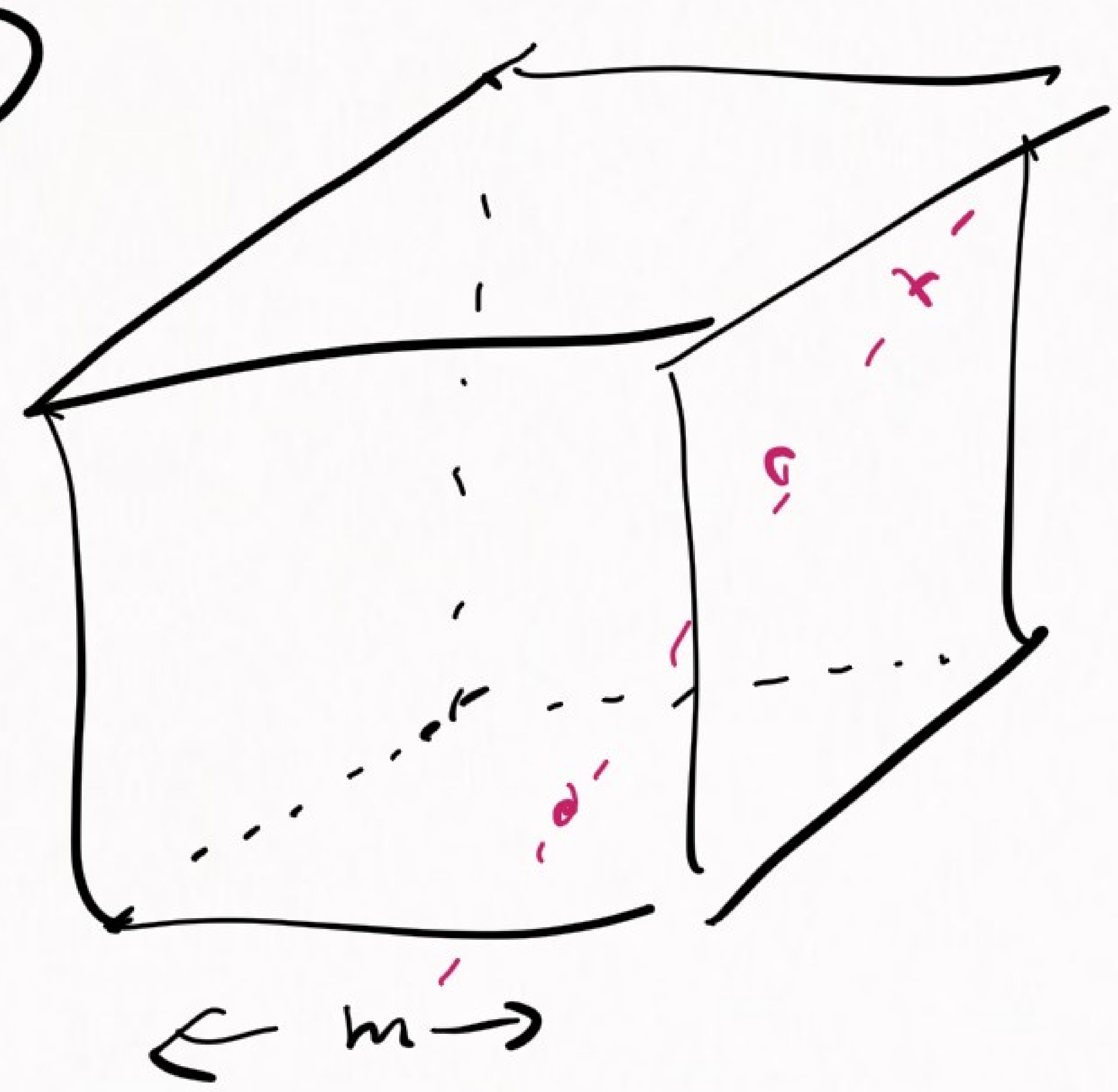
i.e. $\exists$ a col. triple.

Idea start w/ P
- "few" col. 4-tuple
- every "large" subset has "many" col. triple.

$$P \xrightarrow{\text{random subset}} R$$

R
- # col. 4-tuple $\ll |R|$ $\Rightarrow$ deletion
- every "large" subset contains ONE col. triple

Take $[m]^3 = P$



ex #col. triples

$= \Theta(m^6)$

#col. 4-tuple

$= O(m^{6+o(1)})$

Consider p-random subset $R \subseteq [m]^3$.

Supersaturation $\forall\, 0 < \gamma < \frac{1}{2}$, $\forall S \subseteq [m]^3$ of size $|S| > m^{3-\gamma}$

$\Rightarrow$ # col. 3-tuples in $S > m^{6-4\gamma+o(1)}$.

$\mathcal{H}$ 3-unif $\begin{cases} V(\mathcal{H}) = [m]^3 \\ E(\mathcal{H}) = \text{col. 3-tuples} \end{cases}$

indep set in $\mathcal{H}$ $\Longleftrightarrow$ a set in general position.

(Hypergraph container for here)

$\exists\, \mathcal{C} \subseteq 2^{[m]^3}$ s.t.

- $|\mathcal{C}| \le e^{m^{5/3+o(1)}}$
- $\forall$ set T in $[m]^3$ in general position $\exists\, C \in \mathcal{C}$ s.t. $T \subseteq C$
- $\forall\, C \in \mathcal{C}$, $|C| \le m^{8/3+o(1)}$

supersaturation

Continue pf

$p = m^{-1+o(1)}$

$$\begin{cases} \mathbb{E}(|R|) = p \cdot m^3 \approx m^{2+o(1)} = n \\ \mathbb{E}(\#\text{col. 4-tuple}) = m^{6+o(1)} \cdot p^4 = m^{2+o(1)} \end{cases}$$

$\Rightarrow \mathbb{E}(|R|) \gg$

WTS $\forall\, m^{5/3+o(1)}$-subset in R contain col. triple.

Suppose not $\exists$ an $m^{5/3+o(1)}$-subset with no col. triple

- If not $\Rightarrow \exists\ m^{5/3+o(1)}$ - set T in general position $T \subseteq R$

$$\Downarrow$$

- $T \subseteq C$ for some $C \in \mathcal{C}$

$$\Downarrow$$

- For some $C \in \mathcal{C}$,

$$|R \cap C| \geq |T|$$

$$\mathbb{P}\left(\frac{\text{fixed } C \in \mathcal{C}}{\underset{\sim}{|R \cap C| \geq m^{5/3+o(1)}}} \right) \leq e^{-m^{5/3+o(1)}}$$

$$\Uparrow$$

$$\mathbb{E}(|R \cap C|) = |C| \cdot p \leq m^{8/3+o(1)} \cdot p$$

$$= m^{5/3+o(1)}$$

- union bd on $\mathcal{C}$ $\Rightarrow \mathbb{P}\left(\begin{array}{l} \exists\, C \in \mathcal{C} \text{ s.t.} \\ |R \cap C| \geq m^{5/3+o(1)} \end{array} \right)$

$$\leq e^{-m^{5/3+o(1)}} \cdot |\mathcal{C}| \to 0 \qquad \blacksquare$$