Covering cubes by hyperplanes

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Joint work with Alexander Clifton (Emory).

My collaborator

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(Ph.D student at Emory)

A naive question

The n-dimensional cube Q^n consists of the binary vectors $\{0,1\}^n$.

An affine hyperplane is:

$$\{\vec{x} : a_1x_1 + \cdots + a_nx_n = b\}.$$

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What is the minimum number of affine hyperplanes that cover all the vertices of Q^n?

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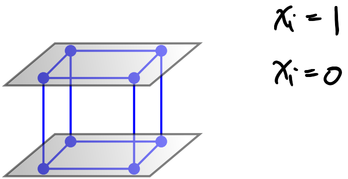
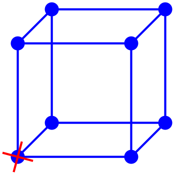


The Alon-Füredi Theorem

A new question

Suppose we would like to avoid exactly one vertex of the cube, how many affine hyperplanes are needed?

For Q^3, 3 planes are needed.



Theorem (Alon, Füredi 1993)

Any set of affine hyperplanes that covers all the vertices of the n-cube Q^n but one contains at least n affine hyperplanes.

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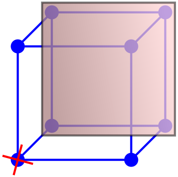

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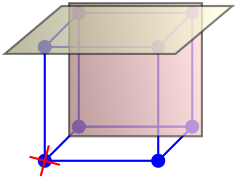

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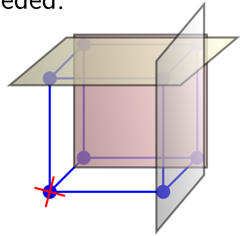

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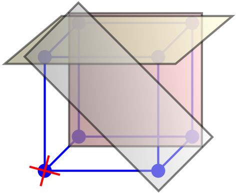

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An outline of the proof of Alon-Füredi Theorem

Proof. Suppose $H_1 \cdots\cdots H_m$ affine hyperplanes covers $Q^n - \{0\}$

$0 \notin H_i$

$H_i: \quad \langle \vec{x}, \vec{a}_i \rangle = b_i \qquad b_i \neq 0$

$\vec{a}_i \in \mathbb{R}^n \quad b_i \in \mathbb{R}$

$$p(x_1 \cdots\cdots x_n) = \prod_{i=1}^{n} (\langle \vec{x}, \vec{a}_i \rangle - b_i)$$

① $p(\vec{x}) = 0$ for

②

Covering the cube twice

Question (Bukh's homework assignment at CMU)

What happens if we would like to cover the vertices of Q^n at least twice, with one vertex uncovered?

Covering the cube k times

Denote by $f(n,k)$ the minimum number of affine hyperplanes needed to cover every vertex of Q^n at least k times (except for $\vec{0}$ which is not covered at all).

We call such a cover an almost k-cover of the n-cube.

$$f(n,1) = n.$$

$$f(n,2) = n+1.$$

What is the next?

Upper and lower bounds

$$f(n,k) \le n + \binom{k}{2}$$

Take

$$x_1 = 1, \cdots, x_n = 1,$$

$$x_1 + \cdots + x_n = 1 \quad \text{for } k-1 \text{ times},$$

$$\vdots$$

$$x_1 + \cdots + x_n = k-1 \quad \text{for } 1 \text{ time}.$$

$$f(n,k) \ge n + k - 1$$

Note that removing $k-1$ planes from an almost k-cover still gives an almost 1-cover.

$$k = 3: \quad n + 2 \le f(n,3) \le n + 3.$$

The $k = 3$ case and a natural conjecture

Theorem (H., Clifton 2019)

For $n \geq 2$,

$$f(n, 3) = n + 3.$$

For $n \geq 3$,

$$f(n, 4) \in \{n + 5, n + 6\}.$$

Conjecture (H., Clifton 2019)

For fixed integer $k \geq 1$ and sufficiently large n,

$$f(n, k) = n + \binom{k}{2}.$$

The Nullstellensatz

If $\mathbb{F}$ is an algebraically closed field, and $f, g_1, \cdots, g_m \in \mathbb{F}[x_1, \cdots, x_n]$, where f vanishes over all common zeros of $g_1, \cdots, g_m$, then there exists an integer k, and polynomials $h_1, \cdots, h_m \in \mathbb{F}[x_1, \cdots, x_n]$, such that

$$f^k = \sum_{i=1}^{m} h_i g_i.$$

When $m = n$, and $g_i = \prod_{s \in S_i}(x_i - s)$, for some $S_1, \cdots, S_n \subset \mathbb{F}$, a stronger result holds: there are polynomials $h_1, \cdots, h_n$ with $\deg h_i \le \deg f - \deg g_i$, such that

$$f = \sum_{i=1}^{n} h_i g_i.$$

Punctured Combinatorial Nullstellensatz

We say $\vec{a} = (a_1, \cdots, a_n)$ is a zero of multiplicity t of $f \in \mathbb{F}[x_1, \cdots, x_n]$, if t is the minimum degree of the terms in $f(x_1 + a_1, \cdots, x_n + a_n)$.

For $i = 1, \cdots, n$, let

$$D_i \subset S_i \subset \mathbb{F}. \quad g_i = \prod_{s \in S_i}(x_i - s). \quad \ell_i = \prod_{d \in D_i}(x_i - d).$$

Theorem (Ball, Serra 2009)

If f has a zero of multiplicity at least t at all the common zeros of $g_1, \cdots, g_n$, except at least one point of $D_1 \times \cdots \times D_n$ where it has a zero of multiplicity less than t, then there are polynomials h_τ satisfying $\deg(h_\tau) \le \deg(f) - \sum_{i \in \tau} \deg(g_i)$, and a non-zero polynomial u satisfying $\deg(u) \le \deg(f) - \sum_{i=1}^n (\deg(g_i) - \deg(\ell_i))$, such that

$$f = \sum_{\tau \in T(n,t)} g_{\tau(1)} \cdots g_{\tau(t)} h_\tau + u \prod_{i=1}^n \frac{g_i}{\ell_i}.$$

$T(n,t)$ consists of all non-decreasing sequences of length t on $[n]$.

Outline of our proof using the PCN ($k = 3$)

Follow-up work

The essence of this proof can be summarized in one sentence:

If f has zeroes of multiplicity at least 3 at $\{0,1\}^n \setminus \{0\}$ and $f(0) \neq 0$, then $\deg(f) \geq n+3$.

Theorem (Sauermann, Wigderson 2020)

For $k \geq 2$, the minimum possible degree of a polynomial $f(x_1, \cdots, x_n)$ sucht that it has zeroes of multiplicity at least k at $\{0,1\}^n \setminus \{0\}$ and $f(0) \neq 0$, is $n + 2k - 3$.

Corollary (Sauermann, Wigderson 2020)

For $k \geq 2$, an almost k-cover of Q^n has at least $n + 2k - 3$ hyperplanes.

$f(n,k)$ for fixed n and large k

For small n, $f(n,k) \neq n + \binom{k}{2}$. Actually,

Theorem (H., Clifton 2019)

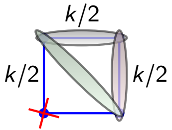
For fixed n, and k tends to infinity,

$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k.$$

- Upper bound: use every hyperplane

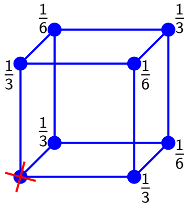
$$x_{i_1} + \cdots + x_{i_j} = 1$$

a total of $\frac{k}{j\binom{n}{j}}$ times. e.g.



$f(n,k)$ for fixed n and large k (ctd.)

- Lower bound: (e.g. $n=3$) assign weights to vertices:



Every affine plane covers vertices of total weight at most 1. Therefore one needs at least

$$k\cdot\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{3}\right)=\frac{11}{6}k$$

hyperplanes.

For general n, assign weight $1/(j\binom{n}{j})$ to vertices whose sum of coordinate is j.

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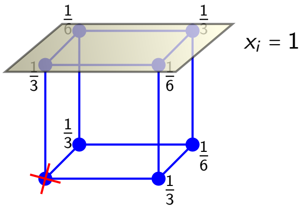

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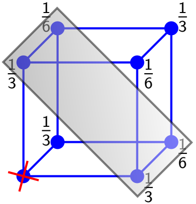

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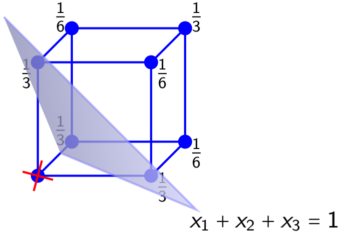

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An LYM-like inenquality

The Lubell-Yamato-Meshalkin inequality

Let $\mathcal{F}$ be a family of subsets in which no set contains another, then

$$\sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1.$$

Lemma (H., Clifton 2019)

Given n real numbers $a_1, \cdots, a_n$, let

$$\mathcal{F} = \{S : \emptyset \neq S \subset [n], \sum_{i \in S} a_i = 1\},$$

then

$$\sum_{S \in \mathcal{F}} \frac{1}{|S| \binom{n}{|S|}} \leq 1.$$

The inequality is tight for all non-zero binary $(a_1, \cdots, a_n)$.

Proof of the Lemma

We associate every $S \in \mathcal{F}$ (binary vector covered by the plane) with some permutations in $\mathcal{P}_S \subset S_n$.

e.g. When $n = 5$, $S = \{1, 3, 4\}$, it means $a_1 + a_3 + a_4 = 1$, take all permutations in S_5 with prefix (i_1, i_2, i_3) satisfying

$$\{i_1, i_2, i_3\} = \{1, 3, 4\}, \quad a_{i_1} < 1, \quad a_{i_1} + a_{i_2} < 1.$$

We can show:

- $\mathcal{P}_S$ are pairwise disjoint.
- $|\mathcal{P}_S| \geq (|S| - 1)!(n - |S|)!$ (the proof uses the *lorry driver puzzle*.)
- Therefore

$$n! \geq \sum_{S \in \mathcal{F}} |\mathcal{P}_S| = \sum_{S \in \mathcal{F}} (|S| - 1)!(n - |S|)!,$$

which simplifies to our desired result.

Future research problems (I)

Problem 1

Prove $f(n,k) = n + \binom{k}{2}$ for large n.

Alon (private communication): for large n, if the almost k-cover contains $x_1 = 1, \cdots, x_n = 1$, then it contains at least $n + \binom{k}{2}$ affine hyperplanes in total.

Problem 2

Let $g(n,m,k)$ be the minimum number of vertices covered less than k times by m affine hyperplanes not passing through $\vec{0}$. Determine $g(n,m,k)$.

Alon, Füredi 1993: $g(n,m,1) = 2^{n-m}$.

Future research problems (II)

Question: Is it true that for all n, m, k:

$$g(n, m, k) = 2^{n-d},$$

where d is the maximum integer such that $f(d, k) \leq m$?

PROBLEM 3

Does there exist an absolute constant $C > 0$, which does not depend on n, such that for a fixed integer n, there exists M_n, so that whenever $k \geq M_n$,

$$f(n, k) \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k + C?$$

Thank you!