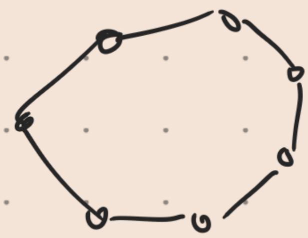
Cycles, minors & expanders.

Def:



#vxs

$n \to \infty$

Fact $\forall$ n-vx graph G has n edges
$\Rightarrow$ $\exists$ a cycle.

Recall ave. deg G : $d(G) = \frac{2e(G)}{n} = \frac{\sum d(v)}{n}$

Rephrase: $d(G) \geq 2 \Rightarrow \exists$ a cycle.
$C(G) \neq \emptyset$.

Def: For a graph G, $C(G)$ = set of cycle lengths in G. $= \{ \ell : \exists C_\ell \subseteq G \}$

Q: Large ave. deg. $\Rightarrow$ say more about $C(G)$?
(constant indep. of n)

B-S: $d(G) \geq n^c \Rightarrow \exists C_{\frac{2}{c}}$.
$0<c<1$

Obs: We cannot force any finite set

Erdős 59 $\exists G \begin{cases} \chi(G) \text{ large} \rightsquigarrow k_1, k_2, \dots, k_i \nearrow \\ \text{girth large} \rightsquigarrow > \max A \end{cases}$

Q: $A \subseteq \mathbb{N}$ finite. $\exists C = C(A)$ s.t.
$\forall G,\ d(G) \geq C \Rightarrow \exists\ C(G) \cap A \neq \phi$

Modified questions: Given large ave. deg.

1) How "dense" is $C(G)$ in $\mathbb{N}$?

2) cycle length in ∞-seq?
(i.e. $C(G) \cap A \neq \phi$? $|A| = \infty$.)

Obs: odds cannot be forced by ave. deg

Consider  $d \to \infty$

$K_{d,d}$,

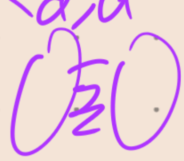
- Erdős – Hajnal 1966

$$\sum_{\ell \in C(G)} \frac{1}{\ell} \quad \text{as a measure for density}$$

Conj $\displaystyle\sum_{\ell \in C(G)} \frac{1}{\ell} \to \infty$ as $\chi(G) \to \infty$

1984 Gyárfás – Komlós – Szemerédi

$$\sum_{\ell \in C(G)} \frac{1}{\ell} = \Omega(\log d(G))$$

Ex $K_{d,d}$  $4, 6, \dots, 2d$ $\quad \sum \frac{1}{\ell} \approx \frac{1}{2} \log d$

Conj Erdős 1975 $\quad \displaystyle\sum \frac{1}{\ell} = \left(\frac{1}{2} + o(1)\right) \log d(G)$.

Conj Erdős-Hajnal 1981

$$\sum_{l \in C_o(G)} \frac{1}{l} \to \infty \quad \text{as} \quad \chi(G) \to \infty$$

odd cycles

Def: $\{\sigma_i\}_{i \in \mathbb{N}}$ unavoidable $\left(\text{w/ } \begin{matrix} d(\cdot) \\ \chi(\cdot) \end{matrix}\right)$

if G has suff. large $\begin{pmatrix} d(\cdot) \\ \chi(\cdot) \end{pmatrix} \Rightarrow \exists$

a cycle w/ length in $\{\sigma_i\}_{i \in \mathbb{N}}$.

ex: $2\mathbb{N}$ is unavoidable w. $d(\cdot)$.

$$d(G) \geq 4 \quad \Rightarrow \quad 2\mathbb{N}$$

1977 Bollobás $\forall$ arith. prog. containing evens

is unavoidable (w/ $d(\cdot)$)

Simplified $AP = \{ak\}_{a \in \mathbb{N}}$

Def Topological minor (subdivision)

<u>Mader 60s</u> $\forall t$, $\exists$ $C(t)$ s.t.

$\forall G$ w/ $d(G) \geq C \Rightarrow \exists$ a subd. of K_t

<u>Erdős 80s asked</u>

$G_i = 2^i$ unavoidable?

$G_i = i^2$?

$G_i =$ primes ± 1 ?

- Sudakov–Verstraëte 08/11.

$d(G) = O(\log_x n)$

Main results (w/ Montgomery 20+)

$\forall G$, $d(G)=d$ (large), $\exists\, \ell \geq d^{1-o(1)}$

s.t. $[\log^{8} \ell, \ell] \subseteq C(G)$

$k=\chi(G)$ $\quad \exists\, \ell : [\ell, \overset{\text{evens}}{\ell \cdot k^{1-o(1)}}] \subseteq C(G)$

<u>Cor</u> 1.) $\sum \frac{1}{\ell} = \left(\frac{1}{2}+o(1)\right) \log d$ ✓

2) $\{2^{i}\}_{i \in \mathbb{N}}$ is unavoidable.

$k=\chi(G)$

<u>Thm</u> 3) $\displaystyle\sum_{\ell \text{ odd}} \frac{1}{\ell} \geq \left(\frac{1}{2}+o(1)\right) \log k$,

- tight, K_k $\quad 3, 5, 7, \dots, k^{\text{odd}}$

Mader $\quad d(G) \geq C(t) \Rightarrow K_t$-subdivision

K-Sz/B-T 90s $\quad d(G) \geq c t^{2} \Rightarrow K_t$-subdivision
$\exists c$

<u>Thomassen 84</u>$^{\text{conj}}$ $\quad d(G) \geq C \Rightarrow$ equally-divide K_t-subd.?

Expander

$|N(X)| = \Omega(|X|) \to \Omega\left(\frac{|X|}{\log^{1+\alpha}|X|}\right)$

K-S$_2$

$\forall G \ \exists H \subseteq G$ s.t. $\begin{cases} \text{- expansion} \\ \text{- } d(H) \geq (1-o(1))\, d(G). \end{cases}$

Idea

even cycle