The Graph Edge Coloring

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Some notations

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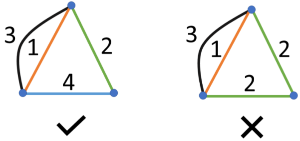

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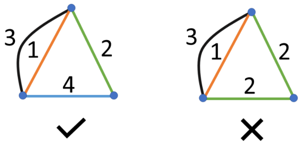

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The earlier work

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- From the preface: "In this book, we survey the literature of the subject of edge-colourings, and describe some more recent results. Many of these results are rather difficult to locate, and some of the most important papers in the field are available only in Russian. With this book, the subject should become more accessible to those interested in the colouring of graphs. "

Citations from the 1978 Fiorini paper

- Fiorini Citation 1
  - Most of the work that has been done since 1964 has been concerned with classification problem.
  - $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ when G is simple.

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- Fiorini Citation 2
  - Although simple graphs has received most attention in the literature, multigraphs have not been entirely ignored.
  - $\chi'(G) \leq \Delta(G) + \mu(G)$ (Vadim G. Vizing 1964)

Book II –Recent work

- Michael Stiebitz, Diego Scheide, Bjarne Toft, Lene Favrholdt Graph edge coloring. – Vizing's theorem and Goldberg's conjecture, John Wiley & Sons, Inc, 2012

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- Publisher's description: "Reviewing recent advances in the Edge Coloring Problem, *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture* provides an overview of the current state of the science, explaining the interconnections among the results obtained from important graph theory studies. The authors introduce many new improved proofs of known results to identify and point to possible solutions for open problems in edge coloring. "

Citation from “Graph edge coloring”

The Edge Color Problem (ECP) is to find the chromatic index $\chi'(G) \dots$. Edge coloring dates back to Peter Guthrie Tait's attempts around 1880 to prove the Four-Color Theorem. Tait observed that coloring the countries of a map (i.e., of a plane, cubic and bridgeless graph G) with four colors is equivalent to coloring the boundaries (i.e., the edges of G) with three colors. Tait claimed that it is easily proved by induction that such a graph G can indeed be edge-colored using three colors, i.e., that $\chi'(G) = 3$. The statement became known as Tait's Theorem, but remained unproved until the Four- Color Theorem was finally resolved. Twenty years after Tait's claim a discussion in a French journal about Tait's Theorem prompted Petersen to present his graph as an example showing that Tait's Theorem is false if the condition that G is planar is omitted.

Topics in Chromatic Graph Theory

- Jessica McDonald Edge-colorings chapter 5

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- Citation from McDonald: The focus of this chapter is on two cornerstones in modern edge-colouring: The famous Goldberg-Seymour conjecture, and ideas culminating in the method of Tashkinov trees. We also discuss extreme examples, the classification problem, and the computational complexity of edge-colouring.

The most recent survey

- Yan Cao, G.C., Guangming Jing, Michael Stiebitz, and Bjarne Toft Graph edge coloring: a survey Graphs and Combinatorics, 2019

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- From Reviewer : This survey provides an overview of the ever-active area of edge coloring, intending to demonstrate techniques of the field to the non-expert.
  - Section 1 includes many major outstanding and recently solved conjectures in the area of edge coloring. These include, most notably, Goldberg's Conjecture (below), Seymour's Exact Conjecture (also below), and the Overfull Conjecture ...
  - Section 2 includes an overview of the techniques used recently to prove Goldberg's Conjecture. These include Vizing's Fans, Kierstead Paths, Tashkinov Trees, and Adjacency Lemmas.

Theorem (Holyer, 1980)

Determining $\chi'(G)$ *is* NP*-hard, even when restricted to a* 3*-regular simple graph.*

maximum degree $\Delta(G)$

A lower bound $\chi'(G) \geq \Delta(G)$

Theorem (Shannon (1949))

$\chi' \leq \frac{3}{2}\Delta$ *for any multigraph* G*.*

Theorem (Vizing (1964), Gupta (1967))

$\chi' \leq \Delta + \mu$ *for any multigraph* G*.*

Remark

$\chi'(G)$ *is* $\Delta(G)$ *or* $\Delta(G) + 1$ *for a simple graph* G*.*

Fact

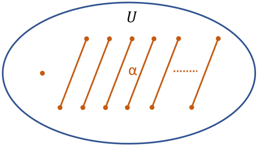
The gap between $\chi'(G)$ and all three bounds in the previous slide can be arbitrarily large:
Let G be a multigraph with three vertices v_1, v_2, and v_3 such that $m(v_1v_2) = k$, $m(v_2v_3) = 2k$, and $m(v_3v_1) = 3k$. Then

- $\chi'(G) = 6k$,
- $\Delta(G) = 5k$,
- $\frac{3}{2}\Delta(G) = 7.5k$, *and*
- $\Delta(G) + \mu(G) = 8k$.

Another Lower Bound - Density

Let φ be a k-edge-coloring of G, U be an odd subset of $V(G)$ and E_α be the set of edges colored by α with both ends in U.

- $|E_\alpha| \le \frac{|U|-1}{2}$.



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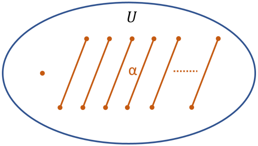

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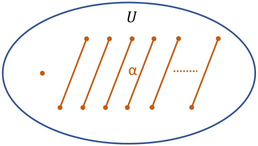

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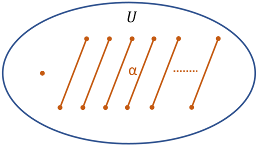

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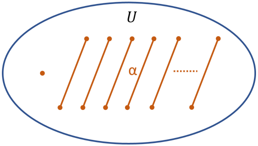

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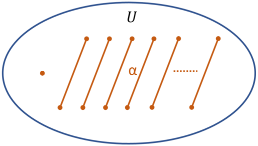

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- $\omega := \omega(G)$ is called the density of G.
- $\chi' \geq \max\{\Delta, \lceil \omega \rceil\}$.

Fractional edge-coloring problem:

Find matchings M_1, M_2, ..., M_t of G and positive numbers y_1, y_2, ..., y_t such that $\sum_{e \in M_i} y_i = 1$ for each edge e and that $\sum_{i=1}^{t} y_i$ is minimized.

Theorem (Seymour, 1979)

Its optimal value, factional chromatic index, $\chi^*(G) = \max\{\Delta(G), \omega(G)\}$.

Theorem (Edmond's Matching Polytope Theorem, 1965)

For any graph $G = (V, E)$, its matching polytope is determined by the system

- $x_e \geq 0, \forall e \in E$ *and* $x(\partial(v)) \leq 1, \forall v \in V$;
- $x(E(U)) \leq \frac{1}{2}(|U| - 1) \forall U \subseteq V$ *with* $|U| \geq 3$ *and odd.*

Fact
$\chi'(G) \geq \chi^*(G) = \max\{\Delta(G), \omega(G)\}$.

Conjecture (Goldberg, 1973, Seymour, 1974)
Every multigraph G satisfies $\chi'(G) \leq \max\{\Delta(G)+1, \lceil \omega(G) \rceil\}$.
i.e., if $\chi'(G) \geq \Delta(G)+2$, *then* $\chi'(G) = \lceil \omega(G) \rceil\}$.

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Remark
The truth of this conjecture would imply the following:

- *There are only two possible values for $\chi'(G)$, so as analogue to Vizing's theorem on edge-coloring of simple graphs, holds for multigraphs.*
- *Every multigraph G satisfies $\chi'(G) - \chi^*(G) \leq 1$, so FECP enjoys a fascinating integer rounding property.*
- *$\chi'(G)$ can be approximated within one of its true value, and hence ECP is one of the "easiest" NP-hard problems.*

Approximate results of the form

$$\chi'(G) \le \max\{\Delta(G) + \rho(G), \lceil \omega(G) \rceil\},$$

where $\rho(G)$ is a positive number depending on G.

- $\rho(G) = o(\Delta(G))$ (Kahn, 1996)

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- $\rho(G) \leq \sqrt[3]{\Delta(G)/4}$ (C., and Jing, 2019)

Approximate results of the form

$$\chi'(G) \leq \max\{\frac{m\Delta(G) + (m-3)}{m-1}, \lceil \omega(G) \rceil\},$$

where m is an odd number.

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- $m = 15$ (Scheide, 2010)
- $m = 23$ (C., Gao, Kim, Postle and Shan, 2018)
- $m = 39$ (C., and Jing, 2019)

Other Results

- $\chi'(G) \le \max\{\Delta(G) + 2\sqrt{\mu(G)\log\Delta(G)}, \lceil\omega(G)\rceil\}$ (Haxell and McDonald, 2012)

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- ...
- Conjecture 1 is true for random multigraphs (Haxell,Krivelevich, and Kronenberg, 2019)

Theorem (C., Jing and Zang, 2018+)

If $\chi'(G) \geq \Delta(G) + 2$, *then* $\chi'(G) = \lceil \omega(G) \rceil$.
(The Goldberg-Seymour conjecture is true.)

r-graph: a multigraph $G = (V, E)$ that is regular of degree r and for every $X \subseteq V$ with $|X|$ odd, the number of edges between X and $V - X$ is at least r.

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Fact

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Theorem (Conjectured by Seymour in 1974)

Every r-graph G satisfies $\chi'(G) \leq r + 1$.

Theorem (Conjectured by Gupta in 1967)

Let G be a multigraph such that $\Delta(G)$ cannot be expressed in the form $2p\mu(G) - q$, where p and q are two integers satisfying $q \geq 0$ and $p > \lfloor (q+1)/2 \rfloor$. Then $\chi'(G) \leq \Delta(G) + \mu(G) - 1$.

Critical multigraph G: $\chi'(H) < \chi'(G)$ for any proper subgraph H of G.

Theorem (Conjectured by Andersen in 1977)

If G is a critical multigraph with $\chi'(G) \geq \Delta(G) + 2$, then $|V|$ is odd.

Theorem (Conjectured by Andersen in 1977)

Let G be a critical multigraph with $\chi'(G) \geq \frac{m\Delta(G)+(m-3)}{m-1}$ for an odd integer $m \geq 3$. Then G has at most $m-2$ vertices.

Definition

The cover index $\xi(G)$ of a multigraph $G = (V, E)$ is the greatest integer k for which there is a coloring of E with k colors, such that each vertex of G is incident with at least one edge of each color.

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Remark

Each color class is an edge cover, which is an edge subset that covers each vertex at least once. So this dual problem is actually the edge cover packing problem.

Recall Co-density of G:

$$\theta(G) = \min\{\frac{2|E^+(U)|}{|U|+1} \ : \ U \subseteq V, |U| \geq 3 \ \text{ and odd}\}$$

where $E^+(G)$ is the set of all edges of G with at least one end in U.

Density of G:

$$\omega(G) = \max\{\frac{|E(U)|}{\lfloor |U|/2 \rfloor} \ : \ U \subset V, \ |U| \geq 3 \text{ odd } \}$$

where $E(U)$ is the set of all edges of G with both ends in U.

Fact

- $\xi(G) \leq \theta(G)$, *because each color class is an edge cover, which contains at least* $(|U|+1)/2$ *edges in* $E^+(U)$.
- $\xi(G) \leq \min\{\delta(G), \lfloor\theta(G)\rfloor\}$

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Conjecture (Gupta, 1978)

Every multigraph G *satisfies* $\xi(G) \ge \min\{\delta(G) - 1, \lfloor\theta(G)\rfloor\}$.

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Every multigraph G *satisfies* $\xi(G) \ge \min\{\delta(G)-1, \lfloor\theta(G)\rfloor\}$.

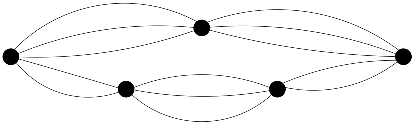
Theorem (Cao, C., Ding, Jing and Zang, 2019+)

For a multigraph G*, there holds*

- $\xi(G) \ge \min\{\delta(G)-1, \lfloor\theta(G)\rfloor\}$ *if* $\theta(G)$ *is not integral and*
- $\xi(G) \ge \min\{\delta(G)-2, \lfloor\theta(G)\rfloor - 1\}$ *otherwise.*

Ring graph

W_t : Ring graph on $2t+1$ vertices.



$\lceil \omega \rceil = \lceil \frac{13}{2} \rceil = 7$, $\Delta = 6$ and $\chi' = \max\{6, 7\}$.

Theorem (Goldberg, 1984)
$\chi'(G) \leq \Delta(G) + 1 + \lfloor \frac{\Delta(G)-2}{g_o(G)-1} \rfloor$.

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Definition

For any integer $\Delta \geq 2$ and an odd integer $g_o \geq 3$, let $\mathcal{GO}(\Delta, g_o)$ denote the set of all graphs G with $\Delta(G) = \Delta$ and odd girth g_o attaining the above bound.

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Conjecture (Stiebitz, Scheide, Toft and Favrholdt, 2012)
Let Δ, g_o be integers such that g_o is odd and $\Delta \ge g_o + 1 \ge 4$. Then every graph $G \in \mathcal{GO}(\Delta, g_o)$ contains as a subgraph a ring graph $G \in \mathcal{GO}(\Delta, g_o)$.

Theorem (Cao, C., He, and Jing, 2019)

The above conjecture fails for every $g_o \geq 5$ *when* $\Delta(G) = \frac{1}{2}(g_0^2 - 2g_0 + 5) - 1$ *and* $\Delta(G) = \frac{1}{2}(g_o^2 - 2g_o + 5) - 2$.

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Theorem (Cao, C., He, and Jing, 2019)

Let Δ, g_o *be integers such that* g_o *is odd,* $\Delta \geq g_o + 1 \geq 4$ *and* $\Delta \geq \frac{1}{2}(g_o^2 - 2g_o + 5)$. *Let* G *be a graph with maximum degree* Δ *and odd girth* g_o. *If* $\chi'(G) = \Delta + 1 + \lfloor \frac{\Delta-2}{g_o-1} \rfloor$, *then* G *contains as a subgraph a ring graph* R *with the same maximum degree and odd girth such that* $\chi'(R) = \Delta + 1 + \lfloor \frac{\Delta-2}{g_o-1} \rfloor$.

Seymour's Exact Conjecture

Conjecture (Seymour's Exact Conjecture)
If G is planar, then $\chi' = \max\{\Delta, \lceil\omega\rceil\}$.

Total Coloring

Definition

A total k-coloring is an assignment of k colors to the vertices and edges of G such that no two adjacent or incident elements of $V(G) \cup E(G)$ receive the same color. The total chromatic number, denoted by $\chi''(G)$, is the minimum $k \geq 0$ such that G admits a total k-coloring.

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Conjecture (Behzad (1965) and Vizing (1968))

$\chi''(G) \leq \Delta(G) + \mu(G) + 1$ *for any graph G. Moreover,* $\chi''(G) \leq \chi'(G) + 1$.

Conjecture (Goldberg (1984))

If $\chi'(G) \geq \Delta(G) + 3$, *then* $\chi''(G) = \chi'(G)$.

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Theorem (Yan Cao, G.C., Guangming Jing (2020+))
If G is a graph satisfying $\chi'(G) \geq \max\{\Delta(G) + 2, |V(G)| + 1\}$, *then* $\chi''(G) = \chi'(G)$.

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If $\chi'(G) \geq \Delta(G) + 2$ *and* G *has a spanning edge-chromatic critical subgraph, then* $\chi''(G) = \chi'(G)$.

Hilton's Overfull conjecture

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If $\Delta > \frac{n}{3}$, *then* $\chi' = \max\{\Delta, \lceil \omega \rceil\}$.

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Theorem (Cao, C., Jing and Shan (19+))
The Hilton-Zhao conjecture is true.

Theorem (Cao, C, and Shan (20+))

Let G be a critical class 2 graph with $\Delta > n/2 + 1$. If there is a Δ-vertex which is adjacent to at most two Δ-vertices, then G is overfull, i.e. $|E(G)|/\lfloor n/2 \rfloor > \Delta$.

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Theorem (Cao, C., and Shan (2020+))

Let G be a critical class 2 graph with $\Delta \geq 3(n-1)/4$. If there is an edge xy such that $d(x) + d(y) = \Delta + 2$, then all other vertices have degree Δ.


Thank You !