Equiangular lines, spherical two-distance sets & spectral graph theory

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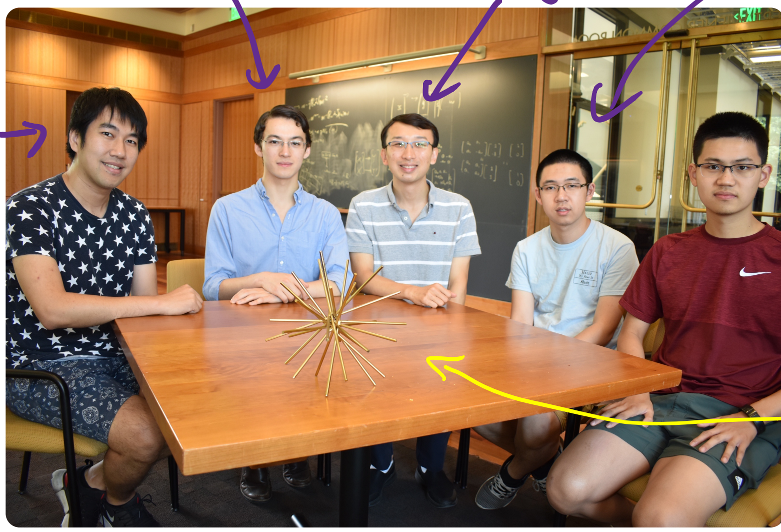


Based on arXiv:1907.12466 & 2006.06633

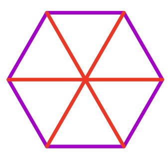
Equiangular lines in $\mathbb{R}^d$ (viewed in $\mathbb{R}^3$)

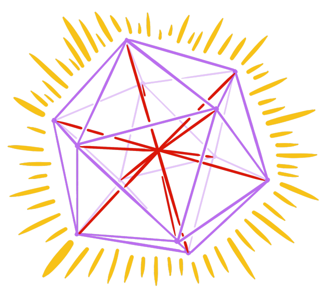
Equiangular lines

$N(d)$ = max # lines in $\mathbb{R}^d$ pairwise same angle

e.g. $N(2)=3$

$N(3)=6$





de Caen '00 $\quad cd^2 \le N(d) \le \binom{d+1}{2}$ $\quad$ Gerzon '73

angles → 90° as $d\to\infty$

Equiangular lines with a fixed angle

$N_\alpha(d) = \max \#$ equiangular lines with angle $\cos^{-1}\alpha$

Some angles are more special

Lemmens-Seidel '73 $\quad N_{1/3}(d) = 2(d-1) \quad \forall d \geq 15$

Neumann '73 $\quad N_\alpha(d) \leq 2d$ unless $\alpha = \frac{1}{\text{odd integer}}$

Neumaier '89 $\quad N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor \quad \forall d \geq d_0$

Next interesting case $\alpha = 1/7$?

Finally, we remark that the recent result of Shearer [13], that every number $t \geqslant t^* = (2+\sqrt{5})^{1/2} \approx 2.058$ is a limit point from above of the set of largest eigenvalues of graphs, makes it likely that the hypothesis of Theorem 2.6 can be satisfied if and only if $t < t^*$. (As communicated to me by Professor J. J. Seidel, Eindhoven, this has indeed been verified by A. J. Hoffman and J. Shearer.) Thus the next interesting case, $t = 3$, will require substantially stronger techniques.

Some decades later ...

Bukh '16 $\quad N_\alpha(d) \leq C_\alpha d$

Balla-Dräxler-Keevash-Sudakov '18 $\quad N_\alpha(d) \leq 1.93d \quad \forall d \geq d_0(\alpha)$ if $\alpha \neq 1/3$

Problem: determine, for each α

$$\lim_{d\to\infty} \frac{N_\alpha(d)}{d}$$

Our work completely solves this problem

Lemmens-Seidel '73: $N_{1/3}(d) = 2(d-1)$ $\forall d$ suff. large

Neumaier '89: $N_{1/5}(d) = \lfloor \frac{3}{2}(d-1) \rfloor$

Our result: $N_{1/7}(d) = \lfloor \frac{4}{3}(d-1) \rfloor$

<u>Thm</u> (JTYZZ) $\forall$ integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

And for other angles $\forall$ fixed $\alpha \in (0,1)$

Set $\lambda = \dfrac{1-\alpha}{2\alpha}$ (reparameterization) $k = k(\lambda)$ "spectral radius order"

Then

$$N_\alpha(d) = \begin{cases} \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor & \forall d \geq d_0(\alpha) \text{ if } k < \infty \\ d + o(d) & \text{if } k = \infty \end{cases}$$

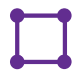
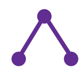
$k(\lambda)$ = spectral radius order

$= \min k$ s.t. $\exists$ k-vertex graph G with $\lambda_1(G) = \lambda$

(set $k(\lambda) = \infty$ if $\nexists$ such G)

spectral radius of G
= top eigenvalue
of adjacency mat of G

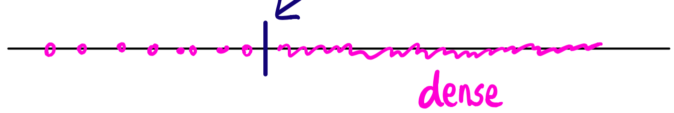
<u>Examples</u>

α	λ	k	G
$1/3$	1	2	
$1/5$	2	3	
$1/7$	3	4	
$1/(1+2\sqrt{2})$	$\sqrt{2}$	3	

$\lim_{d\to\infty} \frac{N_\alpha(d)}{d} = \frac{k(\lambda)}{k(\lambda)-1}$ was conjectured by Jiang – Polyanskii

who proved it for $\lambda < \sqrt{2+\sqrt{5}} \approx 2.058$

$\{\lambda_1(G) : G\} \subset \mathbb{R}$



Hoffman '72 + Shearer '89

Spherical two distance sets

A set of unit vectors in $\mathbb{R}^d$ whose inner products take only two values α, β

(equiangular lines : $\alpha = -\beta$)

Delsarte, Goethals, Seidel '77 max size $\leq \frac{1}{2}d(d+3)$

Taking midpoints of a regular simplex $\rightsquigarrow \frac{1}{2}d(d+1)$

Glazyrin-Yu '18 : tight if $d \geq 7$ & $d+3$ not odd perf sq

From now on let's consider fixed angles

More generally : spherical A-code , $A \subset [-1,1)$

$N_A(d)$ = max # unit vectors in $\mathbb{R}^d$ whose pairwise inner products lie in A

Equiangular lines $\rightsquigarrow$ $A = \{-\alpha, \alpha\}$

We consider $A = \{\alpha, \beta\}$ for fixed $-1 \leq \beta < 0 < \alpha < 1$

Neumaier '81 $\quad N_{\alpha,\beta}(d) \leq 2d+1$ unless $\frac{1-\alpha}{\alpha-\beta} \in \mathbb{Z}$

Bukh '16 $\quad N_{[-1,\beta]\cup\{\alpha\}}(d) = O_\beta(d)$

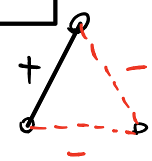
Balla-Dräxler-Keevash-Sudakov '18 $\quad N_{[-1,\beta]\cup\{\alpha_1,\alpha_2,\dots,\alpha_k\}}(d) \leq 2^k (k-1)! \left(1+\frac{\alpha_1}{\beta}+o(1)\right) n^k$

0< < < <

& $\exists\, \alpha_1,\dots,\alpha_k,\beta$ s.t. bound tight up to constant factor

$$\boxed{\textbf{Problem}\ \text{Determine, for fixed } -1 \leq \beta < 0 < \alpha < 1, \quad \lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d}}$$

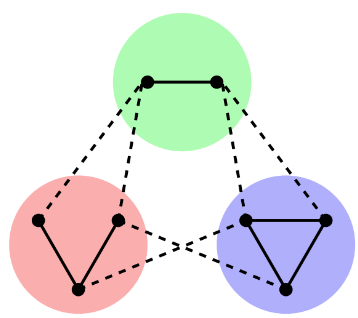
We conjecturally relate this problem to eigenvalues of signed graphs

$G^\pm$ 

$$A_{G^\pm} = \begin{pmatrix} & 1 & -1 \\ 1 & & -1 \\ -1 & -1 & \end{pmatrix}$$

Defn A valid t-coloring:

- $+$ edges join identical colors ———
- $-$ edges join distinct colors - - - - - - -



a valid 3-coloring

$$k_p(\lambda) = \inf\left\{ \frac{|G^\pm|}{\operatorname{mult}(\lambda, G^\pm)} : G^\pm \text{ has valid } p\text{-coloring} \ \& \ \lambda_1(G^\pm) = \lambda \right\}$$

If $G^{\pm}$ has valid 2-coloring
then $G^{\pm}$ is isospectral with its underlying graph

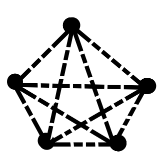
Thus $$k_1(\lambda) = k_2(\lambda) = k(\lambda) = \{|G| : \lambda_1(G) = \lambda\}$$

Determining $k_p(\lambda)$, $p \geq 3$ seems hard

Main conjecture on spherical two-distance sets:
Fix $0 \leq \beta < 0 < \alpha < 1$. Set
$$\lambda = \frac{1-\alpha}{\alpha-\beta} \quad \& \quad p = \left\lfloor \frac{-\alpha}{\beta} \right\rfloor + 1$$
Then
$$\lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d} = \frac{k_p(\lambda)}{k_p(\lambda)-1} \qquad (\ell = 1 \text{ if } k_p(\lambda) = \infty)$$

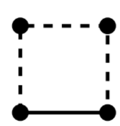
Thm (JTYZZ) Conj true if $p \leq 2$ OR $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$.

$$k_p(1) = \begin{cases} 2 & \text{if } p = 1,2 \\ \frac{p}{p-1} & \text{if } p \geq 2 \end{cases}$$



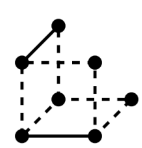
$-4, 1, 1, 1, 1$
$k_5(1) = \frac{5}{4}$

$$k_p(\sqrt{2}) = \begin{cases} 3 & \text{if } p = 1,2 \\ 2 & \text{if } p \geq 2 \end{cases}$$

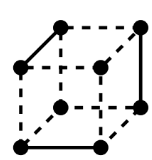


$k_p(\sqrt{2}) = 2$
$\forall p \geq 2$

$$k_p(\sqrt{3}) = \begin{cases} 4 & \text{if } p = 1,2 \\ \frac{7}{3} & \text{if } p = 3 \\ 2 & \text{if } p \geq 4 \end{cases}$$



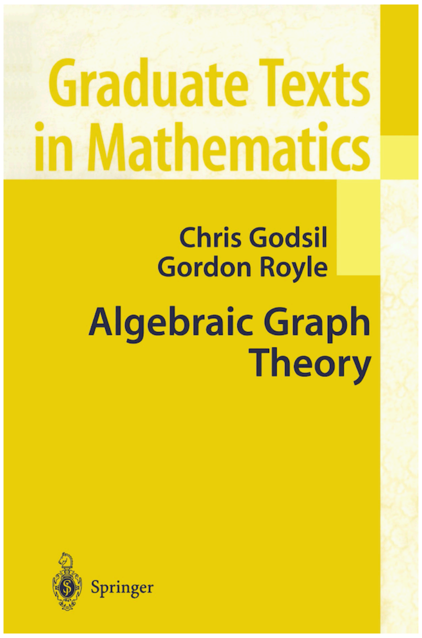
$k_3(\sqrt{3}) = \frac{7}{3}$



$k_p(\sqrt{3}) = 2 \ \forall p \geq 4$

c.f. Huang pf of sensitivity conj

E.g. $(\alpha, \beta) = \left(\frac{2}{5}, \frac{-1}{5}\right)$, $\lambda = 1$, $p = 3$,
$\to k_p(\lambda) = 3$, $\quad N_{\alpha,\beta}(d) = 3d + O(1)$
(contrasting $N_{\alpha,-\alpha}(d) \leq (2+o(1))d$ for eq-ang lines)



The problem that we are about to discuss is one of the founding problems of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A *simplex* in a metric space with distance function d is a subset S such that the distance $d(x, y)$ between any two distinct points of S is the same. In $\mathbb{R}^d$, for example, a simplex contains at most $d+1$ elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of $\mathbb{R}^d$, and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in $\mathbb{R}^d$ such that the angle between any two distinct lines is the same. We call this a set of *equiangular lines*. In this chapter we show how the problem of determining the maximum number of equiangular lines in $\mathbb{R}^d$ can be expressed in graph-theoretic terms.

Unit vectors $v_1, v_2, \ldots, v_N \in \mathbb{R}^d$ $\quad \langle v_i, v_j \rangle = \pm\alpha \ \forall i \neq j$

Associated graph: vtx $[N]$, $i \sim j$ if $\langle v_i, v_j \rangle = -\alpha$ (obtuse)

<u>Observation:</u>

$\exists$ N equiangular lines in $\mathbb{R}^d$ with common angle $\cos^{-1}\alpha$

$\Updownarrow$

$\exists$ N-vtx graph G s.t. $\lambda I - A_G + \frac{1}{2}J$ is positive semidef and rank $\leq d$

$\left(\lambda = \frac{1-\alpha}{2\alpha}\right)$ $\quad$ (J = All 1)

$= \frac{1}{2\alpha}$ Gram matrix

Recall GOAL: maximize N given λ, d

<u>Construction</u>

Starting with some H with $\lambda_1(H) = \lambda$ & $|H| = k(\lambda)$

take $G = H \; H \; H \cdots H \; H$ $\quad$ (N vtx total)

Upper bound on N

$$N = \operatorname{rank}(\lambda I - A_G + \tfrac{1}{2}J) + \operatorname{null}(\lambda I - A_G + \tfrac{1}{2}J)$$
$$\leq d + \operatorname{null}(\lambda I - A_G + \tfrac{1}{2}J)$$
$$\leq d + \underbrace{\operatorname{null}(\lambda I - A_G)}_{= \operatorname{mult}(\lambda, G)} + 1$$

eigenvalue multiplicity

Difficult/interesting case to rule out:

a large connected G with high $\operatorname{mult}(\lambda, G)$

Note that since $\lambda I - A_G + \frac{1}{2}J$ is psd

$\lambda = 1^{st}$ or 2^{nd} largest eigenvalue of A_G

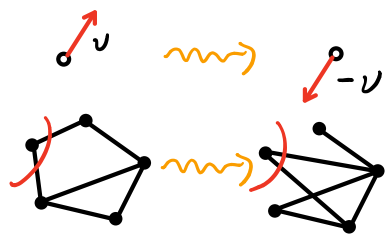
If $\lambda = \lambda_1(G)$, Perron-Frobenius $\Rightarrow \operatorname{mult}(\lambda, G) = 1$

So focus on the case $\lambda = \lambda_2$

> **Q**: Must all connected graphs have small 2^{nd} eigval multiplicity?

Not all graphs can arise from equiangular lines

Switching operation



<u>Thm</u> (Balla–Dräxler–Keevash–Sudakov)
$\forall \alpha\ \exists \Delta$: can switch G to max deg $\leq \Delta$

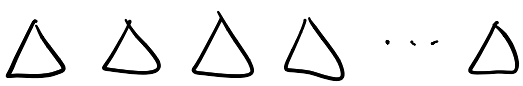
We prove a new result in spectral graph theory:

> <u>Thm</u> [JTYZZ] A connected n-vertex graph with max deg $\leq \Delta$ has second largest eigenvalue with multiplicity $O_\Delta\left(\frac{n}{\log\log n}\right)$

Near miss examples

- Strongly regular graphs — Not bounded degree
  e.g. complete graphs, Paley graphs

-  mult$(0, G)$ linear, 0: a middle eigval

 not connected

<u>Open problem</u> Max. possible 2^{nd} eigval multiplicity of a connected bounded degree graph?

Interesting to consider restrictions to (bdd deg)
- regular graphs
- Cayley graphs

Example: a Cayley graph on $PSL(2,p)$ gives 2^{nd} eigval multiplicity $\gtrsim n^{1/3}$

For expander graphs, $\text{mult}(\lambda_2, G) = O\left(\frac{n}{\log n}\right)$

For non-expanding Cayley graphs, $\text{mult}(\lambda_2, G) = O(1)$

Lee-Makarychev, building on Gromov, Colding-Minicozzi, Kleiner

Recently: McKenzie-Rasmussen-Srivastava
for a connected d-reg graph, $\text{mult}(\lambda_2, G) \le O_d\left(\frac{n}{\log^{\frac{1}{5}-o(1)} n}\right)$

Let's prove:

Thm If G is connected, n vtx, max deg $\le \Delta$ then its 2^{nd} largest eigval has multiplicity $O\left(\frac{n}{\log\log n}\right)$

Lem 1 (Finding a small net)

Every connected n-vtx graph has an r-net of size $\left\lceil \frac{n}{r+1} \right\rceil$ $\forall n, r$

Pf Select a spanning tree

Lem 2 (Net removal significantly reduces spectral radius)

If $H = G - (\text{an } r\text{-net of } G)$

then $\lambda_1(H)^{2r} \le \lambda_1(G)^{2r} - 1$

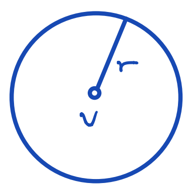
Pf $A_H^{2r} \le A_G^{2r} - I$ entrywise

to check diagonal entries, count closed walks

Suffice to exhibit a closed walk $v \circlearrowleft$ in G not in H

Lem 3 (Local versus global spectra)

$$\sum_{i=1}^{|H|} \lambda_i(H)^{2r} \le \sum_{v \in V(H)} \lambda_1\Big(\underbrace{B_H(v,r)}_{r\text{-neighborhood}}\Big)^{2r}$$



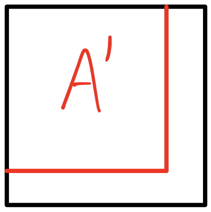
Pf

$\|$

\# closed walks of length $2r$ in H | \# such walks starting at v $= 1_v^T A_{B_H(v,r)}^{2r} 1_v \le \lambda_1(B_H(v,r))^{2r}$ (necessarily stays in $B_H(v,r)$)

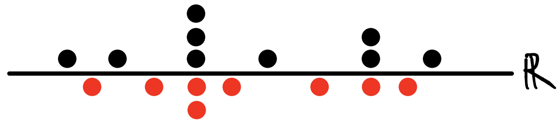
Tool: Cauchy eigenvalue interlacing theorem

Real sym matrix A



remove last row & column $\rightsquigarrow A'$

Then eigenvalues of A & A' interlace



$\Rightarrow$ Deleting a vertex cannot reduce $\text{mult}(\lambda, G)$ by more than 1

Proof sketch that $\text{mult}(\lambda_2, G) = o(n)$

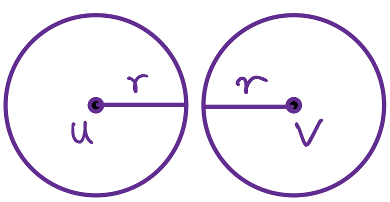
$\lambda = \lambda_2(G) > 0$
(easy if < 0)

Let $r = r_1 + r_2$, $\quad r_1 = c \log\log n$
$\quad r_2 = c \log n$

$$U = \{ v \in V(G) : \lambda_1(B_G(v, r)) > \lambda \}$$

(vtx with large local spectral radius)

If U contains u, v with $d(u,v) \geq 2r + 2$



then G restricted to these two balls has ≥ 2 eigval $> \lambda$. Contradiction.

Thus $U \subset$ a $(2r+1)$-ball $\Rightarrow |U| \leq \Delta^{2r+2} = o(n)$

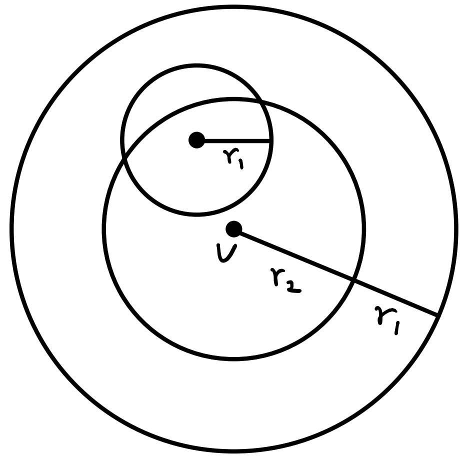
Net removal: let V_0 be an r_1-net of G with $|V_0| \leq \lceil \frac{n}{r_1+1} \rceil$ by Lem 1

Set $H = G - U \cup V_0$

$\forall v \in V(H)$

$B_G(v,r) \setminus B_H(v,r_2)$

is an r_1-net of $B_G(v,r)$



By Lem 2, $\forall v \in V(H)$,

$$\lambda_1(B_H(v,r_2))^{2r_1} \leq \lambda_1(B_G(v,r))^{2r_1} - 1$$
$$\leq \lambda^{2r_1} - 1 \quad (\text{since } v \notin U)$$

By Lem 3,
$$\sum_{i=1}^{|H|} \lambda_i(H)^{2r_2} \leq \sum_{v \in V(H)} \lambda_1(B_H(v,r_2))^{2r_2}$$

$$\mathrm{mult}(\lambda,H)\,\lambda^{2r_2} \leq \sum_{i=1}^{|H|} \lambda_i(H)^{2r_2}, \qquad \sum_{v \in V(H)} \lambda_1(B_H(v,r_2))^{2r_2} \leq (\lambda^{2r_1}-1)^{\frac{r_2}{r_1}} n$$

$$\Rightarrow \mathrm{mult}(\lambda,H) = o(n)$$

By interlacing, $\mathrm{mult}(\lambda,G) \leq \mathrm{mult}(\lambda,H) + |U| + |V| = o(n)$

Summary:

- bound moment by counting closed $2r_2$-walks
- net removal significantly reduces local closed $2r_1$-walks
- relate these via local spectral radii

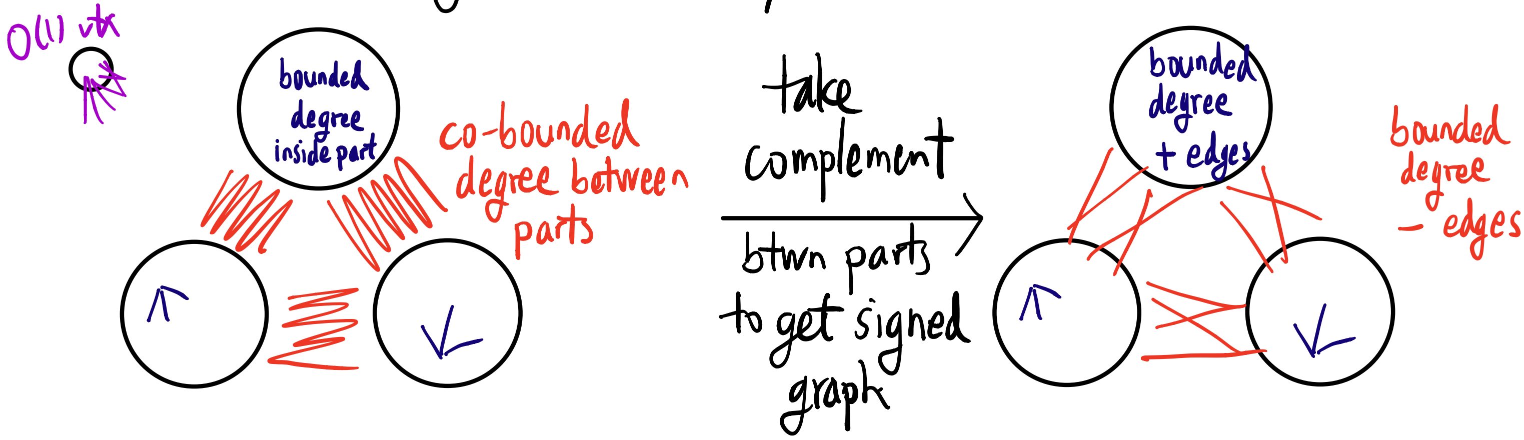
Spherical 2-dist set: unit vectors $v_1, \dots, v_N \in \mathbb{R}^d$
(fixed $-1 < \beta < 0 < \alpha < 1$) $\langle v_i, v_j \rangle \in \{\alpha, \beta\}$ $\forall i \neq j$

Associated graph G: $i \sim j$ if $\langle v_i, v_j \rangle = \beta$ (obtuse)

"Switching" no longer valid

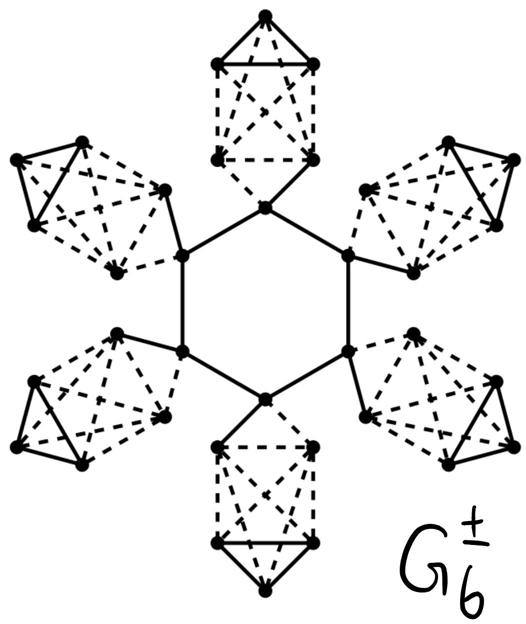
Structure theorem

After deleting $O(1)$ vtx, G can be modified into a complete p-partite graph, $p = \lfloor \frac{-\alpha}{\beta} \rfloor + 1$ where $O(1)$ edges are added/removed at each vtx



We would be able to proceed the same as eq-ang lines if all such signed graphs have $\text{mult}(\lambda_{p+1}, G) = o(n)$

But this is not true



$G_6^\pm$

$\exists\ G_n^\pm$ signed graph
$6n$ vertices
max deg 5
has a valid 3-coloring
BUT: largest eigval appears with multiplicity n

But there is still hope

Our structure theorem actually gives additional forbidden subgraphs not mentioned above

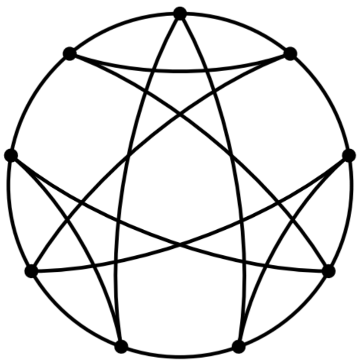
To solve the spherical 2-dist set problem for $\lambda = \frac{1-\alpha}{\alpha-\beta}$ & $p = \lfloor \frac{-\alpha}{\beta} \rfloor + 1$, it suffices to

① Determine

$$k_p(\lambda) = \inf\left\{ \frac{|G^\pm|}{\mathrm{mult}(\lambda, G^\pm)} : G^\pm \text{ has valid } p\text{-coloring } \& \ \lambda_1(G^\pm) = \lambda \right\}$$

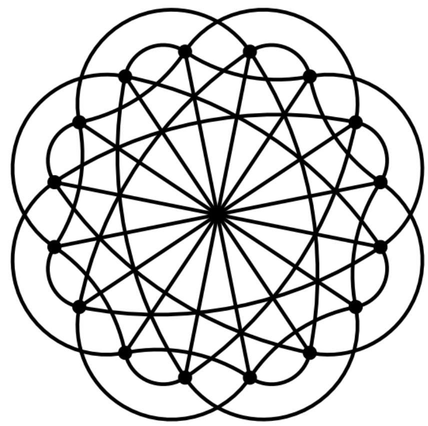
② Give a sufficiently good linear upper bound on $\mathrm{mult}(\lambda_{p+1}, G^\pm)$ for certain classes of signed graphs $G^\pm$ given by forbidden subgraphs

Next interesting case $\lambda = 2$ (we solved $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$)



Paley graph of order 9

$\Rightarrow k_3(2) \leq \frac{9}{4}$



Clebsch graph

$\Rightarrow k_4(2) \leq \frac{8}{5}$