Can the genus of a graph be approximated?

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(joint work with Yifan Jing)

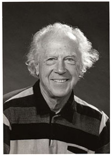
The main result is based on FOCS 2018 talk

Overview

- What is the genus of a graph and why it matters
- Computing the genus (overview)
- Approximation
- Dense case (EPTAS)
- Ingredients
  - Regularity Lemma
  - Hypergraph matching
  - Genus of quasirandom graphs
  - Putting it all together

I. Genus of graphs

G. Ringel and J. W. T. Youngs
Solution of the Heawood map-coloring problem
Proc. Nat. Acad. Sci. U.S.A. (1968)



Percy J. Heawood, Gerhard Ringel, J.W.T. (Ted) Youngs*

Ringel and Youngs determined what is the genus of K_n

*(c) Paul R. Halmos

Map Color Theorem (Ringel and Youngs, 1968)

Conjecture (Heawood, 1890)

For every $g \geq 1$, the maximum chromatic number of a graph that can be embedded in the surface S_g of genus g is

$$\chi(S_g) \leq \left\lfloor \frac{7+\sqrt{48g+1}}{2} \right\rfloor$$

Theorem (Ringel and Youngs, 1968)

$$g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \textit{and} \quad \widetilde{g}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil \quad (n \neq 7).$$

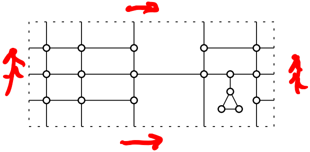
[1] G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A.* (1968)

[2] G. Ringel, Map color theorem. (Springer, 1974)

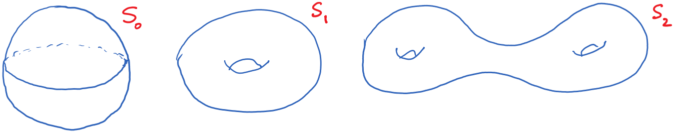
[3] P. J. Heawood, Map-colour theorem. *Quart. J. Pure Appl. Math.* (1890)

The genus problem

Embedding of G: Drawing on a surface without edge-crossings
2-cell embedding: The faces are homeomorphic to (open) disks



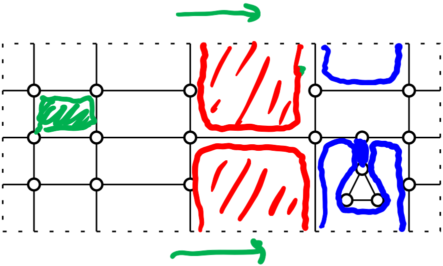
Classification of surfaces: $\mathbb{S}_g$ $(g \geq 0)$ and $\mathbb{N}_h$ $(h \geq 1)$



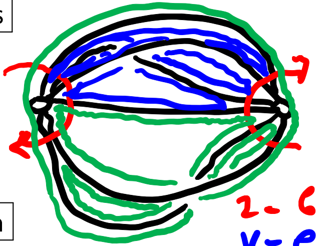
$\mathbb{S}_g$: Orientable surface of genus g

Genus of G: $g(G) = \min\{g \mid G \text{ can be embedded in } \mathbb{S}_g\}$

2-cell embeddings and local rotations



Local rotations



$f = 2$

Euler's formula

$2 - 6 + 2 = 2 - 2g$

$v - e + f = 2 - 2g$

$g = 2$

II. Algorithmic questions

Theorem (Hopcroft and Tarjan / Booth and Luecker 1970's)

It can be decided in linear time if a given graph is planar (genus 0).

Theorem (Kuratowski)

$g(G) > 0$ *if and only if* G *contains* K_5 *or* $K_{3,3}$*-subdivision.*

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Theorem (~~Fillotti, Miller, Reif~~ (1982))

$O(n^{188g})$*-time algorithm to decide if* G *can be embedded in* $\mathbb{S}_g$*.*

Some of their steps may have been oversimplified according to Myrwold (2008).

Garey and Johnson (1979) placed the GENUS PROBLEM on the list of basic problems in NP with unknown hardness.

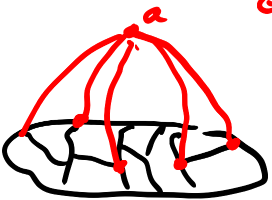
Algorithmic questions

Theorem (Thomassen 1989)

It is NP-hard to compute the genus of (cubic) graphs.

Theorem (M. 2001)

It is NP-hard to compute the genus of apex graphs.



G-a planar

Algorithmic questions

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FPT solutions

Theorem (Robertson and Seymour 1990's)

For every g, it can be decided in cubic time if a given graph has genus at most g.

Theorem (M. 1996)

For every g, it can be decided in linear time if a given graph has genus at most g. Depending on the outcome, an embedding or a forbidden (topological) minor can be found at the same time.

III. Approximating the genus

Can we find an approximation for the genus:

$$\boxed{g(G) \le g \le (1+\varepsilon)g(G)}$$

No constant-factor approximations are known

Known: Factor $c'\sqrt{n}$ and $poly(g)polylog(n)$-approximations.

Different regimes

$$\boxed{g(G) \leq g \leq (1+c)g(G)}$$

There are 4 essentially different ranges where different results occur:

- "Planarly sparse" case (average degree ≤ 6)
  [Conjecture: APX-hard]
- Bounded average degree $6 + \delta < \overline{d}(G) < \Delta$
  [Constant-factor approximation]
- Intermediate average degree
  [Small-constant-factor approximation]
- $\boxed{\text{Dense graphs: } |E(G)| \geq \alpha n^2}$

IV. Genus of dense graphs

Theorem: For dense graphs, $\exists$ EPTAS of time complexity $O_{\varepsilon}(n^2)$

$g(G) \le g \le 1.00001\, g(G)$

$\le (1+\varepsilon)\, g(G)$

IV. Genus of dense graphs

Theorem: For dense graphs, $\exists$ EPTAS of time complexity $O_\varepsilon(n^2)$

$g(G) \leq g \leq 1.00001\, g(G)$

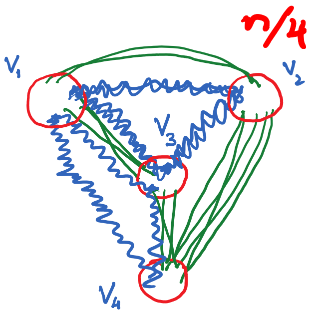
The proof uses:

Szemerédi Regularity Lemma

For every m and $\varepsilon > 0$: $\exists M$ such that every graph of order at least m has equitable partition into k parts for some $m \leq k \leq M$, which is ε-regular.

This means: Parts $V_1, \ldots, V_k$ $(m \leq k \leq M)$ are almost the same in size, all but at most εk^2 pairs of parts (V_i, V_j) are ε-regular (look like random graphs).

Partition



V_1, V_2

$$G = G_0 \cup \left(\bigcup_{\{i,j\}} G_{ij}\right) \cup \left(\bigcup_{\{i,j,\ell\}} G_{ij\ell}\right)$$

$$\|G_0\| < \frac{\varepsilon}{1000}\|G\|$$

G_{ij} quasirandom bipartite

$G_{ij\ell}$ quasirandom tripartite with equal edge densities

$$|V_1| \doteq |V_2| \doteq \cdots \doteq |V_k| \doteq \frac{n}{k}$$

$\bigcup_{\{i,j\}} G_{ij}$ has no triangles

$$d(V_i, V_j) = \frac{e(V_i, V_j)}{(n/4)^2} \doteq p_{ij}$$

Linear program

Goal: use the maximum number of triangles as faces, the rest will be quadrangles

$\mathcal{T}$ triangles *abc* (with positive edge-weights d_e) in the quotient graph

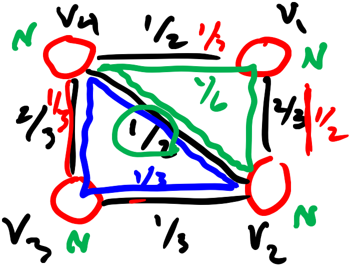
Consider the following LP with indeterminates $\{t(T) \mid T \in \mathcal{T}\}$:

$\nu(H) = \max \sum_{T \in \mathcal{T}} t(T)$
$\sum_{T \ni e, T \in \mathcal{T}} t(T) \leq d_e, \quad \forall e \in E(H)$
$t(T) \geq 0, \quad \forall T \in \mathcal{T}$

G_{14} $\qquad$ G_{124} $1/6$

G_{34} $\qquad$ $G_{234}.$ $1/3$

G_{12}



Main steps of the proof

0. Small graph → Brute force
1. Regularity partition (Frieze-Kannan/Fox et al.)
   G_0; $V_1, V_2, \ldots, V_k$
2. Solve an auxiliary LP to find densities d_{ijl} in order to cover all triangles by quasirandom tripartite graphs $G_{ijl}(V_i, V_j, V_l)$.
3. Find near-quadrangular embeddings of all G_{ij} and near-triangular embeddings of G_{ijl}
4. Combine to obtain a near-optimal embedding of G.

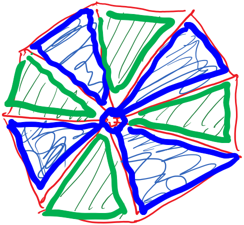
How to find a near-quadrangular (near-triangular) embedding

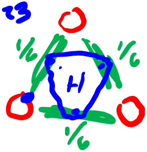
$G = G_{123}$

3-unif.

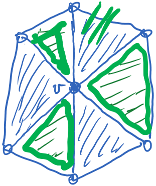
$H = (E(G),\ 3\text{-cycles})$

(3-uniform hypergraph)

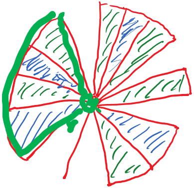




Goal: Find two disjoint almost-perfect matchings in H which are "BLOSSOM–FREE"

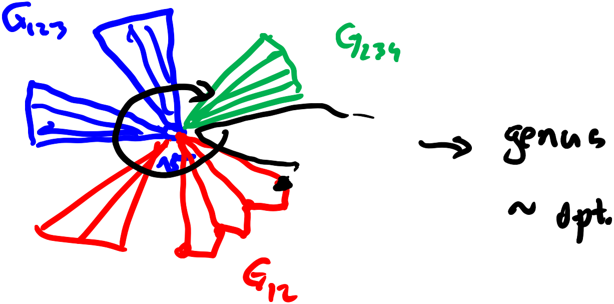


Blossom of length 6



G_{123}

Quasirandom subgraphs G_{ij} and G_{ijl}



Combining triangles and quadrangles from G_{ij} and G_{ijl}

$C \cdot n^2$

$$\frac{\text{R.L.}}{\text{Almost p.m.}} \quad \underline{\underline{\mathcal{H}}} \leftarrow \underline{\underline{LLL}}$$

Thank you for your attention!